ATTRACTORS FOR DAMPED HYPERBOLIC EQUATIONS ON ARBITRARY UNBOUNDED DOMAINS

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ABSTRACT. We prove existence of global attractors for damped hyperbolic equations of the form

$$\varepsilon u_{tt} + \alpha(x)u_t + \beta(x)u - \sum_{ij}(a_{ij}(x)u_{x_j})_{x_i} = f(x,u), \quad x \in \Omega,\, t \in [0,\infty[,$$

$$u(x,t) = 0, \quad x \in \partial\Omega,\, t \in [0,\infty[.$$

on an unbounded domain Ω, without smoothness assumptions on $\beta(\cdot)$, $a_{ij}(\cdot)$, $f(\cdot,u)$ and $\partial\Omega$, and $f(x,\cdot)$ having critical or subcritical growth.

1. INTRODUCTION

In this paper we study the existence of global attractors for semilinear damped wave equations of the form

$$
\begin{aligned}
\varepsilon u_{tt} + \alpha(x)u_t + \beta(x)u - \mathrm{L}u &= f(x,u), \quad x \in \Omega,\, t \in [0,\infty[, \\
u(x,t) &= 0, \quad x \in \partial\Omega,\, t \in [0,\infty[.
\end{aligned}
$$

(1.1)

Here, $N \in \mathbb{N}$ and Ω is an *arbitrary* open set in $\mathbb{R}^N$, bounded or not, $\varepsilon > 0$ is a constant parameter, $\alpha,\, \beta \colon \Omega \to \mathbb{R}$ and $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ are given functions and $\mathrm{L}u := \sum_{ij} \partial_i(a_{ij}(x)\partial_j u)$ is a linear second-order differential operator in divergence form.

For bounded domains Ω there are many results concerning the existence of attractors of (1.1) under various assumptions on ε, α, β, L and f, including the pioneering works by Babin and Vishik [4], Ghidaglia and Temam [11] and Hale and Raugel [14].

The unbounded domain case $\Omega = \mathbb{R}^3$ was considered in the important papers [8, 9] by Feiresl.

In this paper we assume that $\alpha \in L^\infty(\Omega)$, α is bounded below by a positive constant and L is uniformly elliptic with coefficients functions lying in $L^\infty(\Omega)$. We also assume that $\beta \in L^p_u(\mathbb{R}^N)$ with $p > \max(1, N/2)$ and

(1.2) $$\lambda_1 = \inf\{\, E(u) \mid u \in H^1_0(\Omega),\, |u|^2_{L^2(\Omega)} = 1 \,\} > 0$$

where

$$E(u) = \int_\Omega \Big(\sum_{i,j=1}^N a_{ij}(x)\partial_i u(x)\partial_j u(x) + \beta(x)|u(x)|^2 \Big)\,\mathrm{d}x.$$

Here we denote by $L_\mathrm{u}^p(\mathbb{R}^N)$ the set of measurable functions $v\colon \mathbb{R}^N \to \mathbb{R}$ such that

$$|v|_{L_\mathrm{u}^p} := \sup_{y\in\mathbb{R}^N} \left(\int_{B(y)} |v(x)|^p\,\mathrm{d}x \right)^{1/p} < \infty,$$

where, for $y \in \mathbb{R}^N$, $B(y)$ is the open unit cube in $\mathbb{R}^N$ centered at y, cf [3].

We assume that the nonlinearity $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$, $(x,u) \mapsto f(x,u)$ is measurable in x, continuously differentiable in u and satisfies the growth assumptions $f(\cdot,0) \in L^2(\Omega)$ and

$$|\partial_u f(x,u)| \le \overline{C}(a(x) + |u|^{\overline{\rho}})\ \text{for a.e. }\ x \in \Omega\ \text{and every}\ u \in \mathbb{R}.$$

Here $\overline{C} \ge 0$ and $\overline{\rho} \ge 0$ are constants with $2(\overline{\rho}+1) \le 2^* := (2N)/(N-2)$ for $N \ge 3$. If $N \le 2$ or else if $N \ge 3$ and $2(\overline{\rho} + 1) < 2^*$, then $\overline{\rho}$ is called *subcritical*. If $N \ge 3$ and $2(\overline{\rho} + 1) = 2^*$, then $\overline{\rho}$ is called *critical*.

In the subcritical case we also assume that $a \in L_\mathrm{u}^r(\mathbb{R}^N)$ for some $r > \max(N,2)$, while in the critical case we assume that $a \in L^r(\Omega) + L^\infty(\Omega)$ for some $r \ge N$ and $\alpha \in C^1(\Omega)$ with bounded derivatives. (Actually, our assumptions concerning the functions α, β and a are somewhat more general than those listed above.)

Letting $F(x,u) := \int_0^u f(x,s)\,\mathrm{d}s$, $(x,u) \in \Omega \times \mathbb{R}$, we assume the dissipativity conditions
(1.3)
$$f(x,u)u - \overline{\mu}F(x,u) \le c(x)\ \text{and}\ F(x,u) \le c(x)\ \text{for a.e. }\ x \in \Omega\ \text{and every}\ u \in \mathbb{R}$$

where $\overline{\mu} > 0$ is a constant and $c \in L^1(\Omega)$.

The goal of this paper is to prove that under the above hypotheses, Equation (1.1) regarded as a system in (u,v) where $v = u_t$, generates a nonlinear continuous semigroup i.e. a *semiflow* π_f on $Z = H_0^1(\Omega) \times L^2(\Omega)$ which has a global attractor.

Although our results hold for arbitrary open sets Ω, the emphasis here is on unbounded domains.

Condition (1.2) roughly means that the ground state of the stationary Schrödinger equation

$$-\mathrm{L}u + \beta(x)u = 0$$

on Ω with potential β and with Dirichlet boundary condition has positive energy. In the special case of $\beta \in L^1(\Omega)+L^\infty(\Omega)$ with $\beta \ge 0$, condition (1.2) is equivalent to the condition that $\int_G \beta(x)\,\mathrm{d}x = \infty$ for every domain $G \subset \Omega$ that contains arbitrary large balls. This was proved in [1, 2].

The dissipativity condition (1.3) was introduced by Ghidaglia and Temam [11] for the bounded domain case. It is satisfied e.g. if there are constants $\gamma, \nu \in\,]1,\infty[$

and a strictly positive function $D \in L^1(\Omega)$ such that $F(x,u) \leq D(x)$ for all $x \in \Omega$, $u \in \mathbb{R}$ and the function $u \mapsto (\gamma D(x) - F(x,u))^\nu$ is convex for a.e. $x \in \Omega$.

The proofs of our main results are based on Theorem 4.4 below, which provides the so-called *tail estimates* for the solutions $(u(t,x), u_t(t,x))$ of Equation (1.1). For $\overline{\rho}$ subcritical, Theorem 4.4 implies that the semiflow π_f is asymptotically compact on the phase space Z (Lemma 4.9) and this proves the existence of a global attractor in the subcritical case (Theorems 4.10). For $\overline{\rho}$ critical we first use Theorem 4.4 to show that π_f is asymptotically compact with respect to the topology of the space $Y = L^2(\Omega) \times H^{-1}(\Omega)$ (Lemma 4.9). Then we apply a method originally due to J. Ball [5] and elaborated by I. Moise, R. Rosa, X. Wang [18] and G. Raugel [20] to prove that π_f is asymptotically compact on Z (Theorem 4.12). This establishes the existence of a global attractor in the critical case, see Theorem 4.13.

The method of tail estimates was introduced by Wang [22] for parabolic equations on unbounded domains and it was used by Fall and You [10] to establish the existence of an attractor of (1.1) in the special case $\Omega = \mathbb{R}^N$, $\varepsilon = 1$, $\beta(x) \equiv 1$, $\mathrm{L} = \Delta$, $\alpha(x) \equiv \lambda$ with $1 \leq \lambda < 2$, and f dissipative, of *sublinear* growth and having the special form $f(x,u) = g(x) + \phi(u)$ with $g \in L^2(\mathbb{R}^N)$.

We should note that our tail estimates for the solution component $u(t,x)$ do not depend in any way on the finite propagation speed property and are uniform in the parameter $\varepsilon > 0$. This allows us to prove singular semicontinuity results for the family of attractors of Equation (1.1) as $\varepsilon \to 0$, cf. the forthcoming publication [19].

For $N = 3$ the exponent $\overline{\rho}$ is critical if $\overline{\rho} = 2$ and subcritical if $\overline{\rho} < 2$. In particular, Theorem 4.13 extends earlier results by Feireisl [8].

In [9] Feireisl proves existence of attractors even in the supercritical case $2 < \overline{\rho} < 4$. On the other hand, the arguments in [8, 9] require additional smoothness assumptions on $f(x,u)$ with respect to *all* variables (x,u) and some growth assumptions on $|\partial_u f(x,0)|$ and $|\partial_x f(x,0)|$, while we do not need any such condition here. Moreover, only the case $\Omega = \mathbb{R}^3$ and $\mathrm{L} = \Delta$ is considered in [8, 9] and though the proofs do extend to more general domains Ω and to more general differential operators L, restrictions that have to imposed are more stringent than the ones considered here. In fact, the finite propagation speed property used in [8, 9] requires some smoothness assumptions to be imposed on the coefficient functions $a_{ij}(x)$ and on the boundary of Ω, cf. [15], while the Strichartz estimates used in [9] put some additional restrictions both on the shape of Ω and on the coefficient functions $a_{ij}(x)$, cf. [21] and [17].

This paper is organized as follows: in Section 2 we collect some preliminary concepts and results concerning semiflows, attractors and (C_0)-semigroups of linear operators. We also establish an abstract differentiability result, Theorem 2.6, which can frequently be used to rigorously justify formal derivative calculations of functionals along solutions of evolution equations. In Section 3 we establish some general estimates for linear damped wave equations and prove some continuity and differentiability properties of Nemitski operators. Finally, in Section 4, we prove our tail estimates and, as a consequence, establish the existence of a global attractor of Equation (1.1).

Notation. For a and $b \in \mathbb{Z}$ we write $[a..b]$ to denote the set of all $m \in \mathbb{Z}$ with $a \le m \le b$.

Let $N \in \mathbb{N}$ be arbitrary. Given a subset S of $\mathbb{R}^N$ and a function $v \colon S \to \mathbb{R}$ we denote by $\tilde{v} \colon \mathbb{R}^N \to \mathbb{R}$ the trivial extension of v defined by $\tilde{v}(x) = 0$ for $x \in \mathbb{R}^N \setminus S$.

Now let Ω be an arbitrary open set in $\mathbb{R}^N$. Given any measurable function $v \colon \Omega \to \mathbb{R}$ and any $p \in [1, \infty[$ we set, as usual,

$$|v|_{L^p} = |v|_{L^p(\Omega)} := \left(\int_\Omega |v(x)|^p \, \mathrm{d}x \right)^{1/p} \le \infty.$$

Moreover, for $v \in H_0^1(\Omega)$ we set $|v|_{H^1} = |v|_{H^1(\Omega)} := (|\nabla u|_{L^2}^2 + |u|_{L^2}^2)^{1/2}$.

If $k \in \mathbb{N}$ and $f, g \colon \Omega \to \mathbb{R}^k$ are such that $\sum_{i=1}^k f_i g_i \in L^1(\Omega)$ then we set

$$\langle f, g \rangle := \int_\Omega \sum_{i=1}^k f_i(x) g_i(x) \, \mathrm{d}x.$$

We also use the common notation $\mathcal{D}(\Omega)$ resp. $\mathcal{D}'(\Omega)$ to denote the space of all test functions on Ω, resp. all distributions on Ω. If $w \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, then we use the usual functional notation $w(\varphi)$ to denote the value of w at φ.

Given a function $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$, we denote by $\hat{g}$ the (*Nemitski*) operator which associates with every function $u \colon \Omega \to \mathbb{R}$ the function $\hat{g}(u) \colon \Omega \to \mathbb{R}$ defined by

$$\hat{g}(u)(x) = g(x, u(x)), \quad x \in \Omega.$$

All linear spaces considered in this paper are over the real numbers.

2. Preliminaries and an abstract differentiability result

We assume the reader's familiarity with attractor theory on metric spaces as expounded in e.g. [13], [16] or, more recently, in [7] and we just collect here a few relevant concepts from that theory.

Definition. Let X be a metric space. Recall that a *local semiflow* π *on* X is, by definition, a continuous map from an open subset D of $[0, \infty[\times X$ to X such that, for every $x \in X$ there is an $\omega_x = \omega_{\pi,x} \in]0, \infty]$ with the property that $(t, x) \in D$ if and only if $t \in [0, \omega_x[$, and such that (writing $x\pi t := \pi(t, x)$ for $(t, x) \in D$) $x\pi 0 = x$ for $x \in X$ and whenever $(t, x) \in D$ and $(s, x\pi t) \in D$ then $(t + s, x) \in D$ and $x\pi(t + s) = (x\pi t)\pi s$. Given an interval I in $\mathbb{R}$, a map $\sigma \colon I \to X$ is called a *solution (of π)* if whenever $t \in I$ and $s \in [0, \infty[$ are such that $t + s \in I$, then $\sigma(t)\pi s$ is defined and $\sigma(t)\pi s = \sigma(t + s)$. If $I = \mathbb{R}$, then σ is called a *full solution (of π)*. A subset S of X is called *(π-)invariant* if for every $x \in S$ there is a full solution σ with $\sigma(\mathbb{R}) \subset S$ and $\sigma(0) = x$.

Given a local semiflow π on X and a subset N of X, we say that π *does not explode in* N if whenever $x \in X$ and $x\pi[0, \omega_x[\subset N$, then $\omega_x = \infty$. A *global semiflow* is a local semiflow with $\omega_x = \infty$ for all $x \in X$.

Now let π be a global semiflow on X. A subset A of X is called a *global attractor (rel. to π)* if A is compact, invariant and if for every bounded set B in X and every open neighborhood U of A there is a $t_{B,U} \in [0, \infty[$ such that $x\pi t \in U$ for all $x \in B$ and all $t \in [t_{B,U}, \infty[$. It easily follows that a global attractor, if it exists, is uniquely determined.

A subset B of X is called *(π-)ultimately bounded* if there is a $t_B \in [0, \infty[$ such the set $\{\, x\pi t \mid x \in B,\, t \in [t_B, \infty[\,\}$ is bounded.

π is called *asymptotically compact* if whenever $B \subset X$ is ultimately bounded, $(x_n)_n$ is a sequence in B and $(t_n)_n$ is a sequence in $[0, \infty[$ with $t_n \to \infty$ as $n \to \infty$, then the sequence $(x_n \pi t_n)_n$ has a convergent subsequence.

The following result is well-known:

Proposition 2.1. *A global semiflow π on a metric space X has a global attractor if and only if the following conditions are satisfied:*

(1) *π is asymptotically compact;*
(2) *every bounded subset of X is ultimately bounded;*
(3) *there is a bounded set B_0 in X with the property that for every $x \in X$ there is a $t_x \in [0, \infty[$ such that $x\pi t_x \in B_0$.*

Proof. This is just [7, Corollary 1.1.4 and Proposition 1.1.3]. $\square$

We require a few results from the general theory of (C_0)-semigroups of linear operators.

Proposition 2.2. *Let Z be a Banach space and $T(t)$, $t \in [0, \infty[$ be a (C_0)-semigroup of linear operators on Z with generator $B \colon D(B) \to Z$. Then, for every $z \in D(B)$ there is a unique function $u \colon [0, \infty[\to D(B)$ which is continuously differentiable into Z, $u(0) = z$ and*

$$u'(t) = Bu(t), \quad t \in [0, \infty[.$$

u is given by $u(t) = T(t)z$ for all $t \in [0, \infty[$.

Proof. This follows from [12, proof of Theorem II.1.2] $\square$

Proposition 2.3. *Let Z and Y be Banach spaces and $S_Z(t)$, $t \in [0, \infty[$ (resp. $S_Y(t)$, $t \in [0, \infty[$) be a (C_0)-semigroup of linear operators on Z (resp. on Y) with generator $C_Z \colon D(C_Z) \to Z$ (resp. $C_Y \colon D(C_Y) \to Y$). Let $\nu \colon Z \to Y$ be a bounded linear map with $\nu(D(C_Z)) \subset D(C_Y)$. If $\nu C_Z z = C_Y(\nu z)$ for all $z \in D(C_Z)$, then $\nu S_Z(t)z = S_Y(t)(\nu z)$ for all $z \in Z$ and all $t \in [0, \infty[$.*

Proof. An application of Proposition 2.2 shows that $\nu S_Z(t)z = S_Y(t)(\nu z)$ for all $z \in D(C_Z)$ and all $t \in [0, \infty[$. The general case follows by density. $\square$

Proposition 2.4. *Let Z be a Banach space, $S_Z(t)$, $t \in [0, \infty[$ be a (C_0)-semigroup of linear operators on Z with generator $C_Z \colon D(C_Z) \to Z$ and $Q \colon Z \to Z$ be linear*

and bounded. Then the operator $C_Z + Q \colon D(C_Z) \to Z$ generates a (C_0)-semigroup $T_Z(t)$, $t \in [0, \infty[$ of linear operators on Z. Moreover,

$$(2.1) \qquad T_Z(t)z = S_Z(t)z + \int_0^t S_Z(t-s)QT_Z(s)z \, \mathrm{d}s$$

for all $z \in Z$ and $t \in [0, \infty[$.

Proof. The first assertion follows from [12, Theorem I.6.4]. For $z \in D(C_Z) = D(C_Z + Q)$ and $t \in [0, \infty[$ formula (2.1) is proved using Proposition 2.2 and [12, proof of Theorem II.1.3 (ii)]. The general case follows by density. $\square$

Proposition 2.5. *Let Z be a Banach space and $T(t)$, $t \in [0, \infty[$ be a (C_0)-semigroup of linear operators on Z with infinitesimal generator $B \colon D(B) \subset Z \to Z$. Suppose that $\Phi \colon Z \to Z$ is a map which is Lipschitzian on bounded subsets of Z. Then, for each $\zeta \in Z$ there is a maximal $\omega_\zeta = \omega_{B,\Phi,\zeta} \in \,]0, \infty]$ and a uniquely determined continuous map $z = z_\zeta \colon [0, \omega_\zeta[\, \to Z$ such that*

$$(2.2) \qquad z(t) = T(t)\zeta + \int_0^t T(t-s)\Phi(z(s)) \, \mathrm{d}s, \quad t \in [0, \omega_\zeta[.$$

Writing $\zeta \Pi t := z_\zeta(t)$ for $t \in [0, \omega(\zeta)[$ we obtain a local semiflow $\Pi = \Pi_{B,\Phi}$ on Z which does not explode in bounded subsets of Z.

Proof. This follows from [6, proofs of Theorem 4.3.4 and Proposition 4.3.7]. $\square$

In the remaining part of this section we will establish a result which can be used to rigorously justify formal differentiation of various functionals along (mild) solutions of semilinear evolution equations.

Theorem 2.6. *Let Z be a Banach space and $T(t)$, $t \in [0, \infty[$ be a (C_0)-semigroup of linear operators on Z with infinitesimal generator $B \colon D(B) \subset Z \to Z$. Let U be open in Z, Y be a normed space and $V \colon U \to Y$ be a function which, as a map from Z to Y, is continuous at each point of U and Fréchet differentiable at each point of $U \cap D(B)$. Moreover, let $W \colon U \times Z \to Y$ be a function which, as a map from $Z \times Z$ to Y, is continuous and such that $DV(z)(Bz + w) = W(z,w)$ for $z \in U \cap D(B)$ and $w \in Z$. Let $\tau \in \,]0, \infty[$ and $I := [0, \tau]$. Let $\bar{z} \in U$, $g \colon I \to Z$ be continuous and z be a map from I to U such that*

$$z(t) = T(t)\bar{z} + \int_0^t T(t-s)g(s) \, \mathrm{d}s, \quad t \in I.$$

Then the map $V \circ z \colon I \to Y$ is differentiable and

$$(V \circ z)'(t) = W(z(t), g(t)), \quad t \in I.$$

Proof. For $z \in D(B)$ set $|z|_{D(B)} := |z|_Z + |Bz|_Z$. Since B is closed, this defines a complete norm on $D(B)$. For $h \in \,]0, \infty[$ and $t \in I$ set $M_h := \sup_{t \in [0,h]} |T(t)|_{\mathcal{L}(Z,Z)}$ and

$$g_h(t) := (1/h) \int_0^h T(s)g(t) \, \mathrm{d}s.$$

It is well-known that $g_h(t) \in D(B)$ and $Bg_h(t) = (1/h)(T(h)g(t) - g(t))$. Thus $g_h \colon I \to D(B)$ and the estimate

$$
\begin{aligned}
|g_h(t) - g_h(t')|_{D(B)} &= \left|(1/h) \int_0^h T(s)(g(t) - g(t')) \, \mathrm{d}s\right|_Z \\
&\quad + \left|(1/h)(T(h)(g(t) - g(t')) - (g(t) - g(t')))\right|_Z \\
&\leq M_h |g(t) - g(t')|_Z + (1/h)(M_h + 1)|g(t) - g(t')|_Z
\end{aligned}
$$

shows that g_h is continuous into $D(B)$. Moreover, we claim that $g_h(t) \to g(t)$ in Z as $h \to 0^+$, uniformly on I. In fact, otherwise there is an $\varepsilon \in \,]0, \infty[$ and sequences $(h_m)_{m\in\mathbb{N}}$ in $]0, \infty[$ and $(t_m)_{m\in\mathbb{N}}$ in I such that $h_m \to 0$, $t_m \to t \in I$ and $|g_{h_m}(t_m) - g(t_m)|_Z \geq \varepsilon$ for all $m \in \mathbb{N}$. But

$$
|g_{h_m}(t_m) - g(t_m)|_Z \leq |g_{h_m}(t_m) - g(t)|_Z + |g(t_m) - g(t)|_Z.
$$

Moreover,

$$
\begin{aligned}
|g_{h_m}(t_m) - g(t)|_Z &= \left|(1/h_m) \int_0^{h_m} (T(s)g(t_m) - g(t)) \, \mathrm{d}s\right|_Z \\
&\leq \left|(1/h_m) \int_0^{h_m} T(s)(g(t_m) - g(t)) \, \mathrm{d}s\right|_Z + \left|(1/h_m) \int_0^{h_m} (T(s)g(t) - g(t)) \, \mathrm{d}s\right|_Z.
\end{aligned}
$$

W.l.o.g. $h_m \leq 1$ for all $m \in \mathbb{N}$ so

$$
\left|(1/h_m) \int_0^{h_m} T(s)(g(t_m) - g(t)) \, \mathrm{d}s\right|_Z \leq M_1 |g(t_m) - g(t)|_Z \to 0.
$$

Since $T(s)g(t) - g(t) \to 0$ in Z as $s \to 0^+$, it follows that

$$
\left|(1/h_m) \int_0^{h_m} (T(s)g(t) - g(t)) \, \mathrm{d}s\right|_Z \to 0.
$$

Putting things together we see that $|g_{h_m}(t_m) - g(t_m)|_Z \to 0$, a contradiction, proving our claim. Since $D(B)$ is dense in Z there is a sequence $(\bar{z}_m)_{m\in\mathbb{N}}$ in $D(B)$ which converges to $\bar{z}$ in Z. Since U is open in Z we may assume that $\bar{z}_m \in U \cap D(B)$ for all $m \in \mathbb{N}$. Choose a sequence $(h_m)_{m\in\mathbb{N}}$ in $]0, \infty[$ converging to zero. For $m \in \mathbb{N}$ and $t \in I$ set

$$
z_m(t) = T(t)\bar{z}_m + \int_0^t T(t - s)g_{h_m}(s) \, \mathrm{d}s.
$$

It is well-known that $z_m(t) \in D(B)$. Moreover, the map $z_m \colon I \to D(B)$ is continuous into $D(B)$ and differentiable into Z with $z_m'(t) = Bz_m(t) + g_{h_m}(t)$ for $t \in I$. Furthermore, by what we have proved so far, $z_m(t) \to z(t)$ in Z as $m \to \infty$, uniformly on I. It follows that $z_m(t) \in U \cap D(B)$ for some $m_0 \in \mathbb{N}$ and all $m \geq m_0$ and $t \in I$. Moreover, by our hypotheses and by what we have proved so far, $(V \circ z_m)(t) \to (V \circ z)(t)$ and $(V \circ z_m)'(t) = DV(z_m(t))(Bz_m(t) + g_{h_m}(t)) = W(z_m(t), g_{h_m}(t)) \to W(z(t), g(t))$ in Y uniformly on I. Thus $V \circ z$ is differentiable into Y and $(V \circ z)'(t) = W(z(t), g(t))$ for $t \in I$. The theorem is proved. $\quad\square$

3. DAMPED HYPERBOLIC EQUATIONS

For the rest of this paper, $N \in \mathbb{N}$ and Ω is an arbitrary open subset of $\mathbb{R}^N$, bounded or not.

Consider the following

Hypothesis 3.1.

(1) a_0, $a_1 \in]0, \infty[$ are constants and $a_{ij} \colon \Omega \to \mathbb{R}$, i, $j \in [1 .. N]$ are functions in $L^\infty(\Omega)$ such that $a_{ij} = a_{ji}$, i, $j \in [1 .. N]$, and for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, $a_0 |\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \le a_1|\xi|^2$. $A(x) := (a_{ij}(x))_{i,j=1}^{N}$, $x \in \Omega$.

(2) $\beta \colon \Omega \to \mathbb{R}$ is a measurable function with the property that

   (i) for every $\overline{\varepsilon} \in]0, \infty[$ there is a $C_{\overline{\varepsilon}} \in [0, \infty[$ with $\left| |\beta|^{1/2} u \right|_{L^2}^2 \le \overline{\varepsilon}|u|_{H^1}^2 + C_{\overline{\varepsilon}}|u|_{L^2}^2$ for all $u \in H_0^1(\Omega)$;

   (ii) $\lambda_1 := \inf\{\, \langle A\nabla u, \nabla u \rangle + \langle \beta u, u \rangle \mid u \in H_0^1(\Omega),\ |u|_{L^2} = 1 \,\} > 0$.

Remark. Note that, under Hypothesis 3.1 item (1), $\langle A\nabla u, \nabla u \rangle$ is defined and under Hypothesis 3.1 item (2i), $\langle \beta u, u \rangle$ is defined.

The following lemma contains a condition ensuring that β satisfies Hypothesis 3.1 item (2i).

Lemma 3.2. *Let $p \in]1, \infty[$ and $\beta \colon \Omega \to \mathbb{R}$ be such that $\tilde{\beta} \in L_{\mathrm{u}}^p(\mathbb{R}^N)$.*

(1) *If $p \ge N/2$, then there is a $C \in [0, \infty[$ such that*

$$\left| |\beta|^{1/2} u \right|_{L^2} \le C|u|_{H^1}$$

for all $u \in H_0^1(\Omega)$.

(2) *If $p > N/2$, then for every $\overline{\varepsilon} \in]0, \infty[$ there is a $C_{\overline{\varepsilon}} \in [0, \infty[$ with*

$$\left| |\beta|^{1/2} u \right|_{L^2}^2 \le \overline{\varepsilon}|u|_{H^1}^2 + C_{\overline{\varepsilon}}|u|_{L^2}^2$$

for all $u \in H_0^1(\Omega)$.

Proof. There is a family $(y_j)_{j \in \mathbb{N}}$ of points in $\mathbb{R}^N$ such that $\mathbb{R}^N = \bigcup_{j \in \mathbb{N}} \overline{B(y_j)}$ and the sets $\overline{B(y_j)}$, $j \in \mathbb{N}$, are pairwise non-overlapping. Write $B_j = B(y_j)$, $j \in \mathbb{N}$. Let $p' = p/(p-1)$. Since $p \ge N/2$ we have $2p' \le 2^*$ for $N \ge 3$. Let $M \in]0, \infty[$ be a bound of the imbedding $H^1(B) \to L^{2p'}(B)$ where $B = B(0)$. Then, by translation, M is also a bound of the imbedding $H^1(B(y)) \to L^{2p'}(B(y))$ for any $y \in \mathbb{R}^N$. Let

$u \in H_0^1(\Omega)$ be arbitrary. Then

$$\int_\Omega |\beta(x)u^2(x)|\,\mathrm{d}x = \int_{\mathbb{R}^N} |\tilde\beta(x)\tilde u^2(x)|\,\mathrm{d}x = \sum_{j\in\mathbb{N}} \int_{B_j} |\tilde\beta(x)\tilde u^2(x)|\,\mathrm{d}x$$

$$\leq \sum_{j\in\mathbb{N}} \left(\int_{B_j} |\tilde\beta(x)|^p\,\mathrm{d}x\right)^{1/p} \left(\int_{B_j} |\tilde u(x)|^{2p'}\,\mathrm{d}x\right)^{1/p'}$$

$$\leq |\tilde\beta|_{L_\mathrm{u}^p} \sum_{j\in\mathbb{N}} \left(\int_{B_j} |\tilde u(x)|^{2p'}\,\mathrm{d}x\right)^{1/p'} \leq |\tilde\beta|_{L_\mathrm{u}^p} M^2 \sum_{j\in\mathbb{N}} |\tilde u|_{B_j}\big|_{H^1(B_j)}^2$$

$$= |\tilde\beta|_{L_\mathrm{u}^p} M^2 \sum_{j\in\mathbb{N}} \int_{B_j} (|\nabla\tilde u(x)|^2 + |\tilde u(x)|^2)\,\mathrm{d}x = M^2 |\tilde\beta|_{L_\mathrm{u}^p} |\tilde u|_{H^1(\mathbb{R}^N)}^2$$

$$= M^2 |\tilde\beta|_{L_\mathrm{u}^p} |u|_{H^1(\Omega)}^2.$$

This proves the first part of lemma. If $p > N/2$ we may choose q such that $2p' < q$, and $q < 2^*$ for $N \geq 3$. We may then interpolate between 2 and q and so, for every $\overline\varepsilon \in\,]0,\infty[$ there is a $C_{\overline\varepsilon} \in [0,\infty[$, independent of u such that for all $j \in \mathbb{N}$

$$\left(\int_{B_j} |\tilde u(x)|^{2p'}\,\mathrm{d}x\right)^{1/2p'} \leq \overline\varepsilon \left(\int_{B_j} |\tilde u(x)|^q\,\mathrm{d}x\right)^{1/q} + C_{\overline\varepsilon} \left(\int_{B_j} |\tilde u(x)|^2\,\mathrm{d}x\right)^{1/2}$$

$$\leq \overline\varepsilon M' |\tilde u|_{B_j}\big|_{H^1(B_j)} + C_{\overline\varepsilon} |\tilde u|_{B_j}\big|_{L^2(B_j)}.$$

Here $M' \in\,]0,\infty[$ is a bound of the imbedding $H_0^1(B(y_j)) \to L^q(B(y_j))$ for every $j \in \mathbb{N}$. Hence

$$\left(\int_{B_j} |\tilde u(x)|^{2p'}\,\mathrm{d}x\right)^{1/p'} \leq 2(\overline\varepsilon M')^2 \int_{B_j} (|\nabla\tilde u(x)|^2 + |\tilde u(x)|^2)\,\mathrm{d}x + 2C_{\overline\varepsilon}^2 \int_{B_j} |\tilde u(x)|^2\,\mathrm{d}x.$$

Thus, by the above computation,

$$\int_\Omega |\beta(x)u^2(x)|\,\mathrm{d}x \leq |\tilde\beta|_{L_\mathrm{u}^p} \sum_{j\in\mathbb{N}} \left(\int_{B_j} |\tilde u(x)|^{2p'}\,\mathrm{d}x\right)^{1/p'}$$

$$\leq |\tilde\beta|_{L_\mathrm{u}^p} \sum_{j\in\mathbb{N}} \left(2(\overline\varepsilon M')^2 \int_{B_j} (|\nabla\tilde u(x)|^2 + |\tilde u(x)|^2)\,\mathrm{d}x + 2C_{\overline\varepsilon}^2 \int_{B_j} |\tilde u(x)|^2\,\mathrm{d}x\right)$$

$$= |\beta|_{L_\mathrm{u}^p} 2(\overline\varepsilon M')^2 |u|_{H^1}^2 + |\beta|_{L_\mathrm{u}^p} 2C_{\overline\varepsilon}^2 |u|_{L^2}^2.$$

Now an obvious change of notation completes the proof of the second part of the lemma. $\square$

Remark 3.3. Under Hypothesis 3.1 item (1) let the operator L: $H_0^1(\Omega) \to \mathcal{D}'(\Omega)$ be defined by

$$Lu = \sum_{i,j=1}^N \partial_i(a_{ij}\partial_j u), \quad u \in H_0^1(\Omega).$$

The definition of distributional derivatives implies that

(3.1) $$(Lu - \beta u)(v) = -\langle A\nabla u, \nabla v\rangle - \langle \beta u, v\rangle, \quad u \in H_0^1(\Omega), \, v \in \mathcal{D}(\Omega).$$

It follows by density that

(3.2)
$$\langle (Lu - \beta u), v\rangle = -\langle A\nabla u, \nabla v\rangle - \langle \beta u, v\rangle$$
$$\text{for } u, \, v \in H_0^1(\Omega) \text{ with } Lu - \beta u \in L^2(\Omega).$$

Lemma 3.4. *Assume Hypothesis* 3.1. *If* $\kappa \in [0, \lambda_1[$ *is arbitrary and if* $\overline{\varepsilon}$ *and* ρ *are chosen such that* $\overline{\varepsilon} \in \,]0, a_0[$, $\rho \in \,]0, 1[$ *and* $c := \min\big(\rho(a_0 - \overline{\varepsilon}), (1-\rho)(\lambda_1 - \kappa) - \rho(\overline{\varepsilon} + C_{\overline{\varepsilon}} + \kappa)\big) > 0$ *then*

$$c(|\nabla u|_{L^2}^2 + |u|_{L^2}^2) \leq \langle A\nabla u, \nabla u\rangle + \langle \beta u, u\rangle - \kappa\langle u, u\rangle \leq C(|\nabla u|_{L^2}^2 + |u|_{L^2}^2), \; u \in H_0^1(\Omega)$$

where $C := \max(a_1 + \overline{\varepsilon}, \overline{\varepsilon} + C_{\overline{\varepsilon}})$.

Proof. This is just a simple computation. $\square$

Lemma 3.5. *Assume Hypothesis* 3.1 *and let* $\varepsilon \in \,]0, \infty[$ *be arbitrary. For* $u, v \in H_0^1(\Omega)$ *define*

(3.3) $$\langle u, v\rangle_1 = (1/\varepsilon)\langle A\nabla u, \nabla v\rangle + (1/\varepsilon)\langle \beta u, v\rangle.$$

$\langle \cdot, \cdot\rangle_1$ *is a scalar product on* $H_0^1(\Omega)$ *and the norm defined by this scalar product is equivalent to the usual norm on* $H_0^1(\Omega)$.

For every $u \in H_0^1(\Omega)$ *the distribution* $-(1/\varepsilon)Lu + (1/\varepsilon)\beta u \in \mathcal{D}'(\Omega)$ *can be uniquely extended to a continuous linear function* f_u *from* $H_0^1(\Omega)$ *to* $\mathbb{R}$. *The operator*

$$\Lambda: H_0^1(\Omega) \to H^{-1}(\Omega) := (H_0^1(\Omega))', \quad u \mapsto f_u$$

is an isomorphism of normed spaces. The assignment

$$(f, g) \in H^{-1}(\Omega) \times H^{-1}(\Omega) \mapsto \langle f, g\rangle_{-1} := \langle \Lambda^{-1}(f), \Lambda^{-1}(g)\rangle_1$$

defines a scalar product on $H^{-1}(\Omega)$. *The norm defined by this scalar product is equivalent to the usual (operator) norm on* $H^{-1}(\Omega)$.

Proof. This follows from Lemma 3.4 and the Lax-Milgram theorem. $\square$

Proposition 3.6. *Assume Hypothesis 3.1 and let α_0, $\alpha_1 \in [0, \infty[$ and $\varepsilon \in]0, \infty[$ be arbitrary. Let $\alpha \colon \Omega \to \mathbb{R}$ be a measurable function with $\alpha_0 \leq \alpha(x) \leq \alpha_1$ for a.e. $x \in \Omega$. Set $Z = H_0^1(\Omega) \times L^2(\Omega)$ and endow Z with the usual norm $|z|_Z$ defined by*

$$|z|_Z^2 = |\nabla z_1|_{L^2}^2 + |z_1|_{L^2}^2 + |z_2|_{L^2}^2, \quad z = (z_1, z_2).$$

Define $D(B) = D(B_{\alpha, \beta, \varepsilon})$ to be the set of all $(z_1, z_2) \in Z$ such that $z_2 \in H_0^1(\Omega)$ and $\mathrm{L}z_1 - \beta z_1$ (in the distributional sense) lies in $L^2(\Omega)$. Let $B = B_{\alpha, \beta, \varepsilon} \colon D(B) \to Z$ be defined by

$$B(z_1, z_2) = (z_2, -(1/\varepsilon)\alpha z_2 - (1/\varepsilon)\beta z_1 + (1/\varepsilon)\mathrm{L}z_1), \quad z = (z_1, z_2) \in D(B).$$

Under these hypotheses, B is the generator of a (C_0)-semigroup $T(t) = T_{\alpha, \beta, \varepsilon}(t)$, $t \in [0, \infty[$ on Z. If, in addition, $\alpha_0 > 0$, then there are real constants $M = M(\alpha_0, \alpha_1, \varepsilon, \lambda_1) > 0$, $\mu = \mu(\alpha_0, \alpha_1, \varepsilon, \lambda_1) > 0$ such that

$$(3.4) \qquad |T(t)z|_Z \leq Me^{-\mu t}|z|_Z, \quad z \in Z, \, t \in [0, \infty[.$$

Proof. On Z define the scalar product

$$(3.5) \qquad \langle\!\langle (u_1, u_2), (w_1, w_2) \rangle\!\rangle = \langle u_1, w_1 \rangle_1 + \langle u_2, w_2 \rangle_{L^2}.$$

It follows from Lemma 3.5 that the norm $\|(u_1, u_2)\| = \langle\!\langle (u_1, u_2), (u_1, u_2) \rangle\!\rangle^{1/2}$ is equivalent to the norm $|(u_1, u_2)|_Z$.

Now, for $(z_1, z_2) \in D(B)$, we obtain using (3.2)

$$\langle\!\langle B(z_1, z_2), (z_1, z_2) \rangle\!\rangle =$$
$$\langle z_2, z_1 \rangle_1 + \langle -(1/\varepsilon)\alpha z_2 - (1/\varepsilon)\beta z_1 + (1/\varepsilon)\mathrm{L}z_1, z_2 \rangle = -(1/\varepsilon)\langle \alpha z_2, z_2 \rangle.$$

Thus B is dissipative by [6, Proposition 2.4.2]. Let us now show that B is m-dissipative. We use [6, Proposition 2.2.6] and so we only need to show that for every $\lambda \in]0, \infty[$ and for every $(f, g) \in Z$ there is a $(z_1, z_2) \in D(B)$ with

$$(3.6) \qquad (z_1, z_2) - \lambda B(z_1, z_2) = (f, g)$$

Now (3.6) is equivalent to the validity of the two equations

$$(3.7) \qquad z_2 = (1/\lambda)(z_1 - f)$$

and

$$(3.8) \qquad ((1/\lambda) + (1/\varepsilon)\alpha + (1/\varepsilon)\lambda\beta)z_1 - (1/\varepsilon)\lambda\mathrm{L}z_1 = g + ((1/\lambda) + (1/\varepsilon)\alpha)f.$$

Lemma 3.4 and the Lax-Milgram theorem (cf [6, proof of Proposition 2.6.1]) imply that equation (3.8) can be solved for $z_1 \in H_0^1(\Omega)$ with $\mathrm{L}z_1 - \beta z_1 \in L^2(\Omega)$. Now equation (3.7) can be solved for $z_2 \in H_0^1(\Omega)$. It follows that, indeed, B is m-dissipative

and so, by the Hille-Yosida-Phillips theorem, B generates a (C_0)-semigroup $T(t)$, $t \in [0, \infty[$, of linear operators on Z.

Now suppose $\alpha_0 > 0$. Choose μ such that

$$(3.9) \qquad 0 < 2\mu \le \min(1, \alpha_0/(2\varepsilon), \lambda_1/(\varepsilon + \alpha_1)).$$

We now prove that for every $(u_1, u_2) \in Z$

$$(3.10) \qquad \|T(t)(u_1, u_2)\| \le 2e^{-\mu t}\|(u_1, u_2)\|, \quad t \in [0, \infty[.$$

This proves (3.4) in view of the equivalences of the two above norms on Z. By density, it is sufficient to prove (3.10) for $(u_1, u_2) \in D(B)$. Therefore, let $(u_1, u_2) \in D(B)$ be arbitrary and define $(z_1(t), z_2(t)) = T(t)(u_1, u_2)$, $t \in [0, \infty[$. Then the map $t \mapsto z(t) = (z_1(t), z_2(t))$ is differentiable into Z, $z(t) \in D(B)$ and $\dot{z}(t) = Bz(t)$ for $t \in [0, \infty[$. For $t \in [0, \infty[$ let

$$(3.11)$$
$$w(t) = 4\mu\langle z_1(t), z_2(t)\rangle + \langle z_2(t), z_2(t)\rangle$$
$$+ 2(1/\varepsilon)\mu\langle\alpha z_1(t), z_1(t)\rangle + (1/\varepsilon)\langle\beta z_1(t), z_1(t)\rangle + (1/\varepsilon)\langle A\nabla z_1(t), \nabla z_1(t)\rangle.$$

It follows that w is differentiable and a simple calculation shows

$$(3.12)$$
$$\begin{aligned}(1/2)\dot{w}(t) &= \langle(2\mu - (1/\varepsilon)\alpha)z_2(t), z_2(t)\rangle \\ &\quad - 2\mu(1/\varepsilon)\langle\beta z_1(t), z_1(t)\rangle - 2(1/\varepsilon)\mu\langle A\nabla z_1(t), \nabla z_1(t)\rangle \\ &\le \langle(2\mu - (1/\varepsilon)\alpha_0)z_2(t), z_2(t)\rangle \\ &\quad - 2\mu(1/\varepsilon)\langle\beta z_1(t), z_1(t)\rangle - 2(1/\varepsilon)\mu\langle A\nabla z_1(t), \nabla z_1(t)\rangle\end{aligned}$$

By (3.11)

$$\begin{aligned}w(t) &\le 4\mu((1/2)\langle z_1(t), z_1(t)\rangle + (1/2)\langle z_2(t), z_2(t)\rangle) + \langle z_2(t), z_2(t)\rangle \\ &\quad + 2(1/\varepsilon)\mu\langle\alpha z_1(t), z_1(t)\rangle \\ &\quad + (1/\varepsilon)\langle\beta z_1(t), z_1(t)\rangle + (1/\varepsilon)\langle A\nabla z_1(t), \nabla z_1(t)\rangle \\ &\le (2\mu + 1)\langle z_2(t), z_2(t)\rangle + 2\mu(1 + (1/\varepsilon)\alpha_1)\langle z_1(t), z_1(t)\rangle \\ &\quad + (1/\varepsilon)\langle\beta z_1(t), z_1(t)\rangle + (1/\varepsilon)\langle A\nabla z_1(t), \nabla z_1(t)\rangle.\end{aligned}$$

Now

$$\begin{aligned}2\|z(t)\|^2 &= 2\langle z_2(t), z_2(t)\rangle + (1/\varepsilon)\langle\beta z_1(t), z_1(t)\rangle + (1/\varepsilon)\langle A\nabla z_1(t), \nabla z_1(t)\rangle \\ &\quad + (1/\varepsilon)\langle\beta z_1(t), z_1(t)\rangle + (1/\varepsilon)\langle A\nabla z_1(t), \nabla z_1(t)\rangle\end{aligned}$$

By (3.9)

$$2\langle z_2(t), z_2(t)\rangle \ge (2\mu + 1)\langle z_2(t), z_2(t)\rangle$$

and

$$\begin{aligned}(1/\varepsilon)\langle\beta z_1(t), z_1(t)\rangle &+ (1/\varepsilon)\langle A\nabla z_1(t), \nabla z_1(t)\rangle \\ &\ge (1/\varepsilon)\lambda_1\langle z_1(t), z_1(t)\rangle \ge 2\mu(1 + (1/\varepsilon)\alpha_1)\langle z_1(t), z_1(t)\rangle.\end{aligned}$$

Putting things together we see that

$$(3.13) \qquad w(t) \le 2\|z(t)\|^2, \quad t \in [0, \infty[.$$

Moreover, by (3.9)

$$\begin{aligned} w(t) &\ge -4\mu((1/2)4\mu\langle z_1(t), z_1(t)\rangle + (1/2)(1/4\mu)\langle z_2(t), z_2(t)\rangle) + \langle z_2(t), z_2(t)\rangle \\ &\quad + 2(1/\varepsilon)\mu\langle \alpha z_1(t), z_1(t)\rangle + (1/\varepsilon)\langle \beta z_1(t), z_1(t)\rangle + (1/\varepsilon)\langle A\nabla z_1(t), \nabla z_1(t)\rangle \\ &\ge (1/2)\langle z_2(t), z_2(t)\rangle + 2\mu((1/\varepsilon)\alpha_0 - 4\mu)\langle z_1(t), z_1(t)\rangle \\ &\quad + (1/\varepsilon)\langle \beta z_1(t), z_1(t)\rangle + (1/\varepsilon)\langle A\nabla z_1(t), \nabla z_1(t)\rangle \ge (1/2)\|z(t)\|^2. \end{aligned}$$

Thus

$$(3.14) \qquad w(t) \ge (1/2)\|z(t)\|^2, \quad t \in [0, \infty[.$$

By (3.13), (3.12) and (3.9)

$$\begin{aligned} \mu w(t) &\le 2\mu\|z(t)\|^2 = 2\mu\langle z_2(t), z_2(t)\rangle + 2\mu(1/\varepsilon)\langle \beta z_1(t), z_1(t)\rangle \\ &\quad + 2(1/\varepsilon)\mu\langle A\nabla z_1(t), \nabla z_1(t)\rangle \le ((1/\varepsilon)\alpha_0 - 2\mu)\langle z_2(t), z_2(t)\rangle \\ &\quad + 2\mu(1/\varepsilon)\langle \beta z_1(t), z_1(t)\rangle + 2(1/\varepsilon)\mu\langle A\nabla z_1(t), \nabla z_1(t)\rangle \le -(1/2)\dot{w}(t) \end{aligned}$$

so

$$(3.15) \qquad \dot{w}(t) \le -2\mu w(t), \quad t \in [0, \infty[.$$

(3.13), (3.14) and (3.15) imply that

$$\|z(t)\|^2 \le 4e^{-2\mu t}\|z(0)\|^2, \quad t \in [0, \infty[$$

and this in turn implies (3.10). The theorem is proved. $\square$

Proposition 3.7. *Assume Hypothesis 3.1 and let $\varepsilon \in]0, \infty[$ be arbitrary. Define $C_Z := B_{\alpha,\beta,\varepsilon}$ and $S_Z(t) := T_{\alpha,\beta,\varepsilon}(t)$, $t \in [0, \infty[$ with $\alpha \equiv 0$. Moreover, let $Y = L^2(\Omega) \times H^{-1}(\Omega)$ and define the operator $C_Y \colon D(C_Y) \to Y$ by $D(C_Y) = H_0^1(\Omega) \times L^2(\Omega)$ and*

$$C_Y(z_1, z_2) = (z_2, -\Lambda(z_1))$$

where Λ is defined in Lemma 3.5. C_Y is the generator of a (C_0)-semigroup $S_Y(t)$, $t \in [0, \infty[$ of linear operators on Y.

    Finally,

$$\nu S_Z(t)z = S_Y(t)(\nu z), \quad z \in Z, t \in [0, \infty[$$

where $\nu \colon Z \to Y$ is the inclusion map.

Proof. On Y define the scalar product

$$\langle\!\langle (u_1, u_2), (w_1, w_2) \rangle\!\rangle_Y = \langle u_1, w_1 \rangle_{L^2} + \langle u_2, w_2 \rangle_{-1}.$$

It follows from Lemma 3.5 that the norm defined by this scalar product is equivalent to the usual norm on Y. Now, for $(y_1, y_2) \in D(C_Y)$, we easily obtain

$$\langle\!\langle C_Y(y_1, y_2), (y_1, y_2)\rangle\!\rangle_Y = 0.$$

Thus B_Y is dissipative. Using the same arguments as in the proof of Proposition 3.6 (with $\alpha \equiv 0$) we can show that for every $\lambda \in \,]0, \infty[$ and for every $(f, g) \in Y$ there is a $(y_1, y_2) \in D(C_Y)$ with $(y_1, y_2) - \lambda C_Y(y_1, y_2) = (f, g)$. Thus C_Y is m-dissipative and this proves the first assertion. Since, by the definitions of C_Z and C_Y, $\nu D(C_Z) \subset D(C_Y)$ and $\nu C_Z(z_1, z_2) = C_Y \nu(z_1, z_2)$ for all $(z_1, z_2) \in D(C_Z)$, the second assertion follows from Proposition 2.3. $\square$

Proposition 3.8. *Let α, Z and $T(t)$ be as in Proposition 3.6 and Y be as in Proposition 3.7. Suppose that*

(3.16) $$(\exists\, C_1 \in [0, \infty[)(\forall\, z \in L^2(\Omega))\, |\alpha z|_{H^{-1}} \le C_1 |z|_{H^{-1}}.$$

Then there are constants C_2 and $C_3 \in [0, \infty[$ such that

$$|T(t)z|_Y \le C_2 e^{C_3 t}|z|_Y, \quad t \in [0, \infty[,\, z \in Z.$$

Proof. Define the bounded linear map $Q \colon Z \to Z$ by $(z_1, z_2) \mapsto (0, -\alpha z_2)$. By Propositions 2.4 and 2.3 we have, for $z \in Z$ and $t \in [0, \infty[$

$$T(t)z = S_Z(t)z + \int_0^t S_Z(t-s)QT(s)z \,\mathrm{d}s = S_Y(t)z + \int_0^t S_Y(t-s)QT(s)z \,\mathrm{d}s.$$

There are constants C_4 and $C_5 \in [0, \infty[$ such that

$$|S_Y(t)y|_Y \le C_4 e^{C_5 t}|y|_Y, \quad t \in [0, \infty[,\, y \in Y.$$

Using (3.16) we now obtain, for $z \in Z$ and $t \in [0, \infty[$

$$|T(t)z|_Y \le |S_Y(t)z|_Y + \int_0^t |S_Y(t-s)QT(s)z|_Y \,\mathrm{d}s$$
$$\le C_4 e^{C_5 t}|z|_Y + \int_0^t C_4 e^{C_5(t-s)}C_1 |T(s)z|_Y \,\mathrm{d}s.$$

Now Gronwall's lemma completes the proof. $\square$

The next result provides a sufficient condition for the validity of (3.16).

Lemma 3.9. *If $a \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$ and $u \in H_0^1(\Omega)$, then $au \in H_0^1(\Omega)$ and $\partial_i(au) = (\partial_i a)u + a\partial_i u$, $i \in [1..N]$. Moreover, $|au|_{H_0^1} \leq (2N+1)^{1/2}|a|_{W^{1,\infty}}|u|_{H_0^1}$. Furthermore,*

$$|az|_{H^{-1}} \leq (2N+1)^{1/2}|a|_{W^{1,\infty}}|z|_{H^{-1}}, \quad z \in L^2(\Omega).$$

Finally, if U is an open subset of Ω and $a|_U \in C_0^1(U)$ then $(au)|_U \in H_0^1(U)$.

Proof. Set $u_{(i)} = (\partial_i a)u + a\partial_i u$, $i \in [1..N]$. There is a sequence $(v_n)_{n\in\mathbb{N}}$ in $C_0^1(\Omega)$ converging to u in $H^1(\Omega)$. It follows that $av_n \in C_0^1(\Omega)$ and $\partial_i(av_n) = (\partial_i a)v_n + a\partial_i v_n$ for $n \in \mathbb{N}$ and $i \in [1..N]$. Hölder's inequality implies that, for $\varphi \in C_0^1(\Omega)$ and $i \in [1..N]$, $av_n \to au$ and $\partial_i(av_n) \to u_{(i)}$ in $L^2(\Omega)$ while $\varphi\partial_i(av_n) \to \varphi u_{(i)}$ and $(\partial_i\varphi)av_n \to (\partial_i\varphi)au$ in $L^1(\Omega)$. Since $\langle\varphi, \partial_i(av_n)\rangle_{L^2} = -\langle\partial_i\varphi, av_n\rangle_{L^2}$ for $n \in \mathbb{N}$ and $i \in [1..N]$, it follows that $au \in H^1(\Omega)$, $\partial_i(au) = u_{(i)}$ for all $i \in [1..N]$ and

$$(3.17) \qquad\qquad \lim_{n\to\infty} |(av_n) - (au)|_{H^1} = 0.$$

Thus $au \in H_0^1(\Omega)$. This proves the first part of the lemma. It follows that

$$|au|_{H_0^1}^2 = |au|_{L^2}^2 + \sum_{i=1}^N |\partial_i(au)|_{L^2}^2 = |au|_{L^2}^2 + \sum_{i=1}^N |(\partial_i a)u + a\partial_i u|_{L^2}^2$$

$$\leq |a|_{W^{1,\infty}}^2|u|_{L^2}^2 + \sum_{i=1}^N |a|_{W^{1,\infty}}^2(|u|_{L^2} + |\partial_i u|_{L^2})^2$$

$$\leq |a|_{W^{1,\infty}}^2\left(|u|_{L^2}^2 + \sum_{i=1}^N (2|u|_{L^2}^2 + 2|\partial_i u|_{L^2}^2)\right) = |a|_{W^{1,\infty}}^2((2N-1)|u|_{L^2}^2 + 2|u|_{H_0^1}^2)$$

$$\leq |a|_{W^{1,\infty}}^2(2N+1)|u|_{H_0^1}^2.$$

If $z \in L^2(\Omega)$ then $az \in L^2(\Omega)$ and for $v \in H_0^1(\Omega)$ with $|v|_{H^1} \leq 1$ we have $av \in H_0^1(\Omega)$ and

$$|\langle az, v\rangle| = |\langle z, av\rangle| \leq |z|_{H^{-1}}|av|_{H^1} \leq (2N+1)^{1/2}|a|_{W^{1,\infty}}|z|_{H^{-1}}.$$

This proves the second and third part of the lemma. Finally, if $a|_U \in C_0^1(U)$ then $(av_n)|_U \in C_0^1(U)$ for all $n \in \mathbb{N}$ and since, by (3.17), $(av_n)|_U \to (au)|_U$ in $H^1(U)$, it follows that $(au)|_U \in H_0^1(U)$. The lemma is proved $\quad\square$

Proposition 3.10. *Let $a \in C_0^1(\mathbb{R}^N)$ and $r \in [2, \infty[$ be arbitrary. If $N \geq 3$, then assume also that $r < 2^*$. Under these assumptions the map $h: H_0^1(\Omega) \to L^r(\Omega)$, $u \mapsto a|_\Omega \cdot u$, is defined and is linear and compact.*

Proof. There is an open ball U in $\mathbb{R}^N$ such that $\operatorname{supp} a \subset U$. Define the following maps:

$$h_1: H_0^1(\Omega) \to H_0^1(\mathbb{R}^N), \; u \mapsto \tilde{u}, \; h_2: H_0^1(\mathbb{R}^N) \to H_0^1(U), \; v \mapsto (av)|_U,$$
$$h_3: H_0^1(U) \to L^r(U), \; v \mapsto v, \; h_4: L^r(U) \to L^r(\mathbb{R}^N), \; v \mapsto \tilde{v}$$
$$h_5: L^r(\mathbb{R}^N) \to L^r(\Omega), \; v \mapsto v|_\Omega.$$

Clearly, the maps h_1, h_4 and h_5 are defined, linear and bounded, the map h_2 is defined, linear and bounded in view of Lemma 3.9 with $\Omega := \mathbb{R}^N$, while h_3 is defined, linear and compact by Rellich embedding theorem. Since, for all $u \in H_0^1(\Omega)$, $(h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1)(u) = a|_\Omega \cdot u$, it follows that h is defined and $h = h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1$ so h is linear and compact. $\square$

Definition. A function $f: \Omega \times \mathbb{R} \to \mathbb{R}$, $(x, u) \mapsto f(x, u)$ is said to satisfy a C^0-(resp. C^1-)*Carathéodory condition*, if for every $u \in \mathbb{R}$ the partial map $x \mapsto f(x, u)$ is Lebesgue-measurable and for a.e. $x \in \Omega$ the partial map $u \mapsto f(x, u)$ is continuous (resp. continuously differentiable).

If $f: \Omega \times \mathbb{R} \to \mathbb{R}$, $(x, u) \mapsto f(x, u)$ satisfies a C^0-Carathéodory condition, define the function $F: \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$F(x, u) = \int_0^u f(x, s) \, \mathrm{d}s,$$

whenever $s \mapsto f(x, s)$ is continuous and $F(x, u) = 0$ otherwise. F is called the *canonical primitive of f*.

Given $\overline{C}, \overline{\rho} \in [0, \infty[$, a measurable function $a: \Omega \to \mathbb{R}$ and a null set $M \subset \Omega$, a function $g: (\Omega \setminus M) \times \mathbb{R} \to \mathbb{R}$, $(x, u) \mapsto g(x, u)$ is said to satisfy a $(\overline{C}, \overline{\rho}, a)$-*growth condition*, if $|g(x, u)| \leq \overline{C}(|a(x)| + |u|^{\overline{\rho}})$ for every $x \in \Omega \setminus M$ and every $u \in \mathbb{R}$. The number $\overline{\rho}$ is called *subcritical* if $N \leq 2$ or ($N \geq 3$ and $\overline{\rho} < (2^*/2) - 1$). $\overline{\rho}$ is called *critical* if $N \geq 3$ and $\overline{\rho} = (2^*/2) - 1$.

Proposition 3.11. *Let f satisfy a C^1-Carathéodory condition and $\partial_u f$ satisfy a $(\overline{C}, \overline{\rho}, a)$-growth condition. Let F be the canonical primitive of f. Then, for a.e. $x \in \Omega$ and all $u, h \in \mathbb{R}$*

$$(3.18) \qquad |f(x, u) - f(x, 0)| \leq \overline{C}|a(x)||u| + \overline{C}|u|^{\overline{\rho}+1},$$

$$(3.19) \qquad |f(x, u+h) - f(x, u)| \leq \overline{C}|a(x)||h| + \overline{C}\max(1, 2^{\overline{\rho}-1})(|u|^{\overline{\rho}} + |h|^{\overline{\rho}})|h|,$$

$$(3.20) \qquad |F(x, u)| \leq \overline{C}(|a(x)||u|^2/2 + |u|^{\overline{\rho}+2}/(\overline{\rho}+2)) + |u||f(x, 0)|,$$

$$
(3.21) \qquad
\begin{aligned}
&|F(x, u+h) - F(x, u)| \leq \\
&(|f(x, 0)| + \overline{C}|a(x)|(|u| + |h|) + \overline{C}\max(1, 2^{\overline{\rho}})(|u|^{\overline{\rho}+1} + |h|^{\overline{\rho}+1}))|h|,
\end{aligned}
$$

and

$$
(3.22) \qquad
\begin{aligned}
&|F(x, u+h) - F(x, u) - f(x, u)h| \\
&\leq (\overline{C}|a(x)| + \overline{C}\max(1, 2^{\overline{\rho}-1})(|u|^{\overline{\rho}} + |h|^{\overline{\rho}}))|h|^2.
\end{aligned}
$$

Moreover, for every measurable function $v: \Omega \to \mathbb{R}$ both $\hat{f}(v)$ and $\hat{F}(v)$ are measurable and for all measurable functions $u, h: \Omega \to \mathbb{R}$

$$(3.23) \qquad |\hat{f}(u)|_{L^2} \leq |\hat{f}(0)|_{L^2} + \overline{C}(|au|_{L^2} + |u|_{L^{2(\overline{\rho}+1)}}^{\overline{\rho}+1}),$$

(3.24)
$$|\hat{f}(u+h) - \hat{f}(u)|_{L^2}$$
$$\leq \overline{C}|ah|_{L^2} + \overline{C}\max(1, 2^{\overline{\rho}-1})(|u|^{\overline{\rho}}_{L^{2(\overline{\rho}+1)}} + |h|^{\overline{\rho}}_{L^{2(\overline{\rho}+1)}})|h|_{L^{2(\overline{\rho}+1)}},$$

(3.25)
$$|\hat{F}(u)|_{L^1} \leq \overline{C}(|a|u|^2|_{L^1}/2 + |u|^{\overline{\rho}+2}_{L^{\overline{\rho}+2}}/(\overline{\rho}+2)) + |u|_{L^2}|\hat{f}(0)|_{L^2},$$

(3.26)
$$|\hat{F}(u+h) - \hat{F}(u)|_{L^1} \leq$$
$$(|\hat{f}(0)|_{L^2} + \overline{C}(|au|_{L^2} + |ah|_{L^2}) + \overline{C}\max(1, 2^{\overline{\rho}})(|u|^{\overline{\rho}+1}_{L^{2(\overline{\rho}+1)}} + |h|^{\overline{\rho}+1}_{L^{2(\overline{\rho}+1)}}))|h|_{L^2},$$

and

(3.27)
$$|\hat{F}(u+h) - \hat{F}(u) - \hat{f}(u)h|_{L^1}$$
$$\leq (\overline{C}|ah|_{L^2} + \overline{C}\max(1, 2^{\overline{\rho}-1})(|u|^{\overline{\rho}}_{L^{2(\overline{\rho}+1)}} + |h|^{\overline{\rho}}_{L^{2(\overline{\rho}+1)}})|h|_{L^{2(\overline{\rho}+1)}})|h|_{L^2}.$$

Finally, if $\overline{\rho}$ is critical, then for every $r \in [N, \infty[$ there is a constant $C(r) \in [0, \infty[$ such that whenever $a = a_1 + a_2$ with $a_1 \in L^r(\Omega)$ and $a_2 \in L^\infty(\Omega)$, then for all u, $h \in H_0^1(\Omega)$

(3.28)
$$|\hat{f}(u+h) - \hat{f}(u)|_{H^{-1}} \leq C(r)(|a_1|_{L^r} + |a_2|_{L^\infty})|h|_{L^2}$$
$$+ C(r)(|u|^{\overline{\rho}}_{L^{2*}} + |h|^{\overline{\rho}}_{L^{2*}})|h|_{L^2}.$$

Proof. For a.e. $x \in \Omega$ and all $u, h \in \mathbb{R}$ we have

$$f(x, u+h) - f(x, u) = \int_0^1 \partial_u f(x, u + \theta h)h \, d\theta.$$

$$F(x, u+h) - F(x, u) - f(x, u)h = \int_0^1 [f(x, u + \theta h) - f(x, u)]h \, d\theta$$
$$= \int_0^1 [\int_0^1 \partial_u f(x, u + r\theta h)\theta h \, dr]h \, d\theta$$

These equalities and the definition of F imply estimates (3.18), (3.19), (3.20), (3.21) and (3.22). Now well known arguments and Hölder inequality yields the remaining assertions of the proposition except (3.28). To prove (3.28), let $r \in [N, \infty[$ and u, h and $v \in H_0^1(\Omega)$ be arbitrary. Then

$$|\langle \hat{f}(u+h) - \hat{f}(u), v \rangle| \leq \int_\Omega |f(x, (u+h)(x)) - f(x, u(x))| \, |v(x)| \, dx$$

$$+ \overline{C} \int_\Omega |(ah)(x)| \, |v(x)| \, dx + \overline{C}\max(1, 2^{\overline{\rho}-1}) \int_\Omega (|u(x)|^{\overline{\rho}} + |h(x)|^{\overline{\rho}})|h(x)||v(x)| \, dx.$$

Now

$$\int_\Omega |(a_1 h)(x)|\, |v(x)|\, \mathrm{d}x \le |a_1|_{L^r} |h|_{L^2} |v|_{L^{2r/(r-2)}}$$

and

$$\int_\Omega |(a_2 h)(x)|\, |v(x)|\, \mathrm{d}x \le |a_2|_{L^\infty} |h|_{L^2} |v|_{L^2}.$$

Moreover, since $(1/2^*) + (1/2) + (1/N) = 1$ and $N\overline{\rho} = 2^*$ we also have

$$\int_\Omega (|u(x)|^{\overline{\rho}} + |h(x)|^{\overline{\rho}})|h(x)||v(x)|\, \mathrm{d}x \le (|u|_{L^{2^*}}^{\overline{\rho}} + |h|_{L^{2^*}}^{\overline{\rho}})|h|_{L^2}|v|_{L^{2^*}}$$

Noting that $2r/(r-2) \le 2^*$ let C be a common bound of the imbeddings $H_0^1(\Omega) \to L^s(\Omega)$ for $s \in \{2, 2^*, 2r/(r-2)\}$. Then we conclude

$$|\langle \hat{f}(u+h) - \hat{f}(u), v \rangle|$$
$$\le \overline{C}C(|a_1|_{L^r} + |a_2|_{L^\infty})|h|_{L^2}|v|_{H^1} + \overline{C}\max(1, 2^{\overline{\rho}-1})C(|u|_{L^{2^*}}^{\overline{\rho}} + |h|_{L^{2^*}}^{\overline{\rho}})|h|_{L^2}|v|_{H^1}.$$

Since

$$|\hat{f}(u+h) + \hat{f}(u)|_{H^{-1}} = \sup_{v \in H_0^1(\Omega)} |\langle \hat{f}(u+h) - \hat{f}(u), v \rangle|$$

estimate (3.28) follows.   □

Standing Assumption. *For the rest of this paper, we assume Hypothesis 3.1 and fix an $\varepsilon \in\,]0, \infty[$. Let $Z = H_0^1(\Omega) \times L^2(\Omega)$ and $B = B_{\alpha,\beta,\varepsilon}$ be defined as in Proposition 3.6. Moreover, let $Y = L^2(\Omega) \times H^{-1}(\Omega)$.*

Proposition 3.12. *Let $\overline{C}, \overline{\rho} \in [0, \infty[$ and $a\colon \Omega \to \mathbb{R}$ be a measurable function such that the assignments $u \mapsto |a|u$ and $u \mapsto |a|^{1/2}u$ induce bounded linear operators from $H_0^1(\Omega)$ to $L^2(\Omega)$. Suppose the function f satisfies a C^1-Carathéodory condition and $\partial_u f$ satisfies a $(\overline{C}, \overline{\rho}, a)$-growth condition. Moreover, suppose $f(\cdot, 0) \in L^2(\Omega)$. If $N \ge 3$, then assume also that $\overline{\rho} \le (2^*/2) - 1$. Under these hypotheses, f induces a map $\hat{f}\colon H_0^1(\Omega) \to L^2(\Omega)$ which is Lipschitzian on bounded subsets of $H_0^1(\Omega)$. The canonical primitive F of f induces a map $\hat{F}\colon H_0^1(\Omega) \to L^1(\Omega)$. This map is Fréchet differentiable and $D\hat{F}(u)[h] = \hat{f}(u) \cdot h$ for u and $h \in H_0^1(\Omega)$. The map $\Phi_f\colon Z \to Z$,*

$$(3.29) \qquad \Phi_f(z) = (0, (1/\varepsilon)\hat{f}(z_1)), \quad z = (z_1, z_2) \in Z,$$

is bounded and Lipschitzian on bounded subsets of $H_0^1(\Omega)$. By π_f we denote the local semiflow $\Pi_{B,\Phi}$ on Z, where $\Phi = \Phi_f$. This local semiflow does not explode in bounded subsets of $H_0^1(\Omega)$.

Proof. This follows from Proposition 3.11, the Sobolew imbedding theorem, Proposition 3.6 and Proposition 2.5.   □

Remark 3.13. By Lemma 3.2 the hypotheses on the function a imposed in Proposition 3.12 are satisfied e.g. if $\tilde{a} \in L_u^p(\mathbb{R}^N)$ with $p \ge N$.

Remark 3.14. The local semiflow π_f defined in Proposition 3.12 is, by definition, the *local semiflow generated by solutions of the damped wave equation* (1.1).

4. TAIL ESTIMATES AND THE EXISTENCE OF ATTRACTORS

Proposition 4.1. *Let $\overline{\gamma}\colon \mathbb{R}^N \to [0,1]$ be a C^1-function such that $\sup_{x\in\mathbb{R}^N}(|\overline{\gamma}(x)|^2 + |\nabla\overline{\gamma}(x)|^2) < \infty$. Set $\gamma = \overline{\gamma}^2$. Assume the hypotheses and notations of Proposition 3.12. Fix $\delta \in \,]0,\infty[$, and define the functions $V = V_\gamma\colon Z \to \mathbb{R}$ and $V^* = V_\gamma^*\colon Z \to \mathbb{R}$ by*

$$V(z) = (1/2)\int_\Omega \gamma(x)\Psi_z(x)\,\mathrm{d}x$$

and

$$V^*(z) = \int_\Omega \gamma(x)F(x, z_1(x))\,\mathrm{d}x$$

for $z = (z_1, z_2) \in Z$. Here, for $z \in Z$ and $x \in \Omega$,

$$\Psi_z(x) = \varepsilon|\delta z_1(x) + z_2(x)|^2 + (A\nabla z_1)(x) \cdot \nabla z_1(x) + (\beta(x) - \delta\alpha(x) + \delta^2\varepsilon)|z_1(x)|^2.$$

Let $\tau_0 \in \,]0,\infty[$, $I = [0, \tau_0]$ and $z\colon I \to Z$ be a solution of π_f. Then the functions $V \circ z$ and $V^* \circ z$ are differentiable and, for $t \in I$,

$$(V \circ z)'(t) =$$
$$\int_\Omega \gamma(x)\big(\varepsilon(\delta z_1 + z_2)(\delta z_2 + (-(1/\varepsilon)\alpha(x)z_2 + (1/\varepsilon)f(x, z_1(t)(x))))$$
$$+ (-\delta\alpha(x) + \delta^2\varepsilon)z_1 z_2 - \delta\beta(x)z_1 z_1\big)\,\mathrm{d}x - \delta\int_\Omega \gamma(x)(A\nabla(z_1)) \cdot \nabla z_1\,\mathrm{d}x$$
$$- \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma) \cdot \nabla z_1\,\mathrm{d}x$$

$$(V^* \circ z)'(t) = \int_\Omega \gamma(x)f(x, z_1(t)(x))z_2(t)(x)\,\mathrm{d}x.$$

(4.1)
$$(V \circ z)'(t) + 2\delta(V \circ z)(t) = \int_\Omega \gamma(x)(2\delta\varepsilon - \alpha(x))(\delta z_1 + z_2)^2\,\mathrm{d}x$$
$$+ \int_\Omega \gamma(x)(\delta z_1 + z_2)f(x, z_1(t)(x))\,\mathrm{d}x$$
$$- \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma) \cdot \nabla z_1\,\mathrm{d}x$$

Proof. By Proposition 3.11 we have that V and V^* are defined and Fréchet differentiable on Z and for all $z = (z_1, z_2)$ and $\xi = (\xi_1, \xi_2)$ in Z

$$DV(z)[\xi] = \int_\Omega \gamma(x)\big(\varepsilon(\delta z_1(x) + z_2(x))(\delta\xi_1(x) + \xi_2(x))$$
$$+ (A(x)\nabla z_1(x)) \cdot \nabla\xi_1(x) + (\beta(x) - \delta\alpha(x) + \delta^2\varepsilon)z_1(x)\xi_1(x)\big)\,\mathrm{d}x$$

and

$$DV^*(z)[\xi] = \int_\Omega \gamma(x) f(x, z_1(x)) \xi_1(x) \, \mathrm{d}x.$$

In particular, for $z = (z_1, z_2) \in D(B)$ and $w = (w_1, w_2) \in Z$ we obtain, omitting the argument $x \in \Omega$ in some of the expressions below,

$$DV(z)[Bz + w]$$
$$= \int_\Omega \gamma(x) \big(\varepsilon(\delta z_1 + z_2)(\delta(z_2 + w_1) + (-(1/\varepsilon)\alpha(x)z_2 - (1/\varepsilon)\beta(x)z_1 + (1/\varepsilon)\mathrm{L}z_1 + w_2))$$
$$+ (A\nabla z_1) \cdot \nabla(z_2 + w_1) + (\beta(x) - \delta\alpha(x) + \delta^2\varepsilon)z_1(z_2 + w_1) \big) \, \mathrm{d}x$$

and

$$DV^*(z)[Bz + w] = \int_\Omega \gamma(x) f(x, z_1(x))(z_2 + w_1) \, \mathrm{d}x.$$

Evaluating further we see that

$$DV(z)[Bz + w]$$
$$= \int_\Omega \gamma(x) \big(\varepsilon(\delta z_1 + z_2)(\delta(z_2 + w_1) + (-(1/\varepsilon)\alpha(x)z_2 - (1/\varepsilon)\beta(x)z_1 + w_2))$$
$$+ (A\nabla z_1) \cdot \nabla w_1 + (\beta(x) - \delta\alpha(x) + \delta^2\varepsilon)z_1(z_2 + w_1) \big) \, \mathrm{d}x$$
$$+ \int_\Omega \gamma(x) \big((\delta z_1 + z_2)\mathrm{L}z_1 + (A\nabla z_1) \cdot \nabla z_2 \big) \, \mathrm{d}x.$$

By Green's formula

$$\int_\Omega \gamma(x) \big((\delta z_1 + z_2)\mathrm{L}z_1 + (A\nabla z_1) \cdot \nabla z_2 \big) \, \mathrm{d}x = - \int_\Omega \gamma(x)(A\nabla(\delta z_1 + z_2)) \cdot \nabla z_1 \, \mathrm{d}x$$
$$- \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma) \cdot \nabla z_1 \, \mathrm{d}x + \int_\Omega \gamma(x)(A\nabla z_1) \cdot \nabla z_2 \, \mathrm{d}x$$
$$= - \int_\Omega \gamma(x)(A\nabla(\delta z_1)) \cdot \nabla z_1 \, \mathrm{d}x - \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma) \cdot \nabla z_1 \, \mathrm{d}x$$

so we obtain

$$DV(z)[Bz + w]$$
$$= \int_\Omega \gamma(x) \big(\varepsilon(\delta z_1 + z_2)(\delta(z_2 + w_1) + (-(1/\varepsilon)\alpha(x)z_2 + w_2))$$
$$+ (A\nabla z_1) \cdot \nabla w_1 + (-\delta\alpha(x) + \delta^2\varepsilon)z_1(z_2 + w_1) + \beta(x)(z_1 w_1 - \delta z_1 z_1) \big) \, \mathrm{d}x$$
$$- \int_\Omega \gamma(x)(A\nabla(\delta z_1)) \cdot \nabla z_1 \, \mathrm{d}x - \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma) \cdot \nabla z_1 \, \mathrm{d}x.$$

Define the maps $W\colon Z \times Z \to \mathbb{R}$ and $W^*\colon Z \times Z \to \mathbb{R}$ by

$$
\begin{aligned}
W(z,w) = {} & \int_\Omega \gamma(x)\big(\varepsilon(\delta z_1 + z_2)(\delta(z_2 + w_1) + (-(1/\varepsilon)\alpha(x)z_2 + w_2)) \\
& + (A\nabla z_1)\cdot\nabla w_1 + (-\delta\alpha(x) + \delta^2\varepsilon)z_1(z_2 + w_1) + \beta(x)(z_1 w_1 - \delta z_1 z_1)\big)\,\mathrm{d}x \\
& - \int_\Omega \gamma(x)(A\nabla(\delta z_1))\cdot\nabla z_1\,\mathrm{d}x - \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma)\cdot\nabla z_1\,\mathrm{d}x
\end{aligned}
$$

and

$$
W^*(z,w) = \int_\Omega \gamma(x)f(x, z_1(x))(z_2(x) + w_1(x))\,\mathrm{d}x
$$

for $(z,w) \in Z \times Z$. In the particular case where $w_1 = 0$ we thus obtain

(4.2)
$$
\begin{aligned}
W(z,w) = {} & \int_\Omega \gamma(x)\big(\varepsilon(\delta z_1 + z_2)(\delta z_2 + (-(1/\varepsilon)\alpha(x)z_2 + w_2)) \\
& + (-\delta\alpha(x) + \delta^2\varepsilon)z_1 z_2 - \delta\beta(x)z_1 z_1\big)\,\mathrm{d}x \\
& - \int_\Omega \gamma(x)(A\nabla(\delta z_1))\cdot\nabla z_1\,\mathrm{d}x - \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma)\cdot\nabla z_1\,\mathrm{d}x
\end{aligned}
$$

and

(4.3)
$$
W^*(z,w) = \int_\Omega \gamma(x)f(x, z_1(x))z_2(x)\,\mathrm{d}x.
$$

Using Hypothesis 3.1 and Proposition 3.11 we see that W and W^* are continuous from $Z \times Z$ to $\mathbb{R}$ and so Theorem 2.6, formulas (4.2) and (4.3) and a straightforward computation complete the proof. $\square$

Consider the following hypothesis.

Hypothesis 4.2.
(1) $\alpha_0 > 0$;
(2) $\overline{C}, \overline{\rho}, \overline{\tau} \in [0, \infty[$ and $\overline{\mu} \in]0, \infty[$ are constants and $c\colon \Omega \to [0, \infty[$ is a function with $c \in L^1(\Omega)$. If $N \geq 3$, then $\overline{\rho} \leq (2^*/2) - 1$;
(3) $a\colon \Omega \to \mathbb{R}$ is a measurable function such that the assignments $u \mapsto |a|u$ and $u \mapsto |a|^{1/2}u$ induce bounded linear operators from $H_0^1(\Omega)$ to $L^2(\Omega)$;
(4) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a C^1-Carathéodory condition;
(5) F is the canonical primitive of f;
(6) $\partial_u f$ satisfies a $(\overline{C}, \overline{\rho}, a)$-growth condition;
(7) $|f(\cdot, 0)|_{L^2} \leq \overline{\tau}$;
(8) $f(x,u)u - \overline{\mu}F(x,u) \leq c(x)$ and $F(x,u) \leq c(x)$ for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$.

A sufficient condition for the dissipativity assumption (8) to hold is contained in the following lemma:

Lemma 4.3. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy a C^0-Carathéodory condition and F be the canonical primitive of f. Let ν, $\gamma \in \,]1, \infty[$ be constants and $D \in L^1(\Omega)$ be a function with $D(x) > 0$ for all $x \in \Omega$ and such that $F(x, u) \leq D(x)$ for all $x \in \Omega$ and all $u \in \mathbb{R}$. Assume also that the function $u \mapsto (\gamma D(x) - F(x, u))^\nu$ is convex for a.e. $x \in \Omega$.*

Then $f(x, u)u - \overline{\mu} F(x, u) \leq c(x)$ and $F(x, u) \leq c(x)$ for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$. Here, $\overline{\mu} := (1/\nu)$ and $c(x) := \max(1, \gamma^\nu (\gamma - 1)^{1-\nu} \nu^{-1}) D(x)$, $x \in \Omega$.

Proof. Define $G(x, u) = -(\gamma D(x) - F(x, u))^\nu$ for $x \in \Omega$ and $u \in \mathbb{R}$. Our convexity assumption implies that the function $u \mapsto \partial_u G(x, u)$ is nonincreasing and continuous for a.e. $x \in \Omega$. Notice that whenever $h : \mathbb{R} \to \mathbb{R}$ is continuous and nonincreasing then $h(u)u \leq \int_0^u h(s)\,\mathrm{d}s$ for all $u \in \mathbb{R}$. It follows that, for a.e. $x \in \Omega$ and every $u \in \Omega$,

$$
(4.4) \qquad \begin{aligned}
\nu f(x, u)(\gamma D(x) - F(x, u))^{\nu - 1} u &\leq G(x, u) - G(x, 0) \\
&= -(\gamma D(x) - F(x, u))^\nu + (\gamma D(x))^\nu.
\end{aligned}
$$

Since, for a.e. $x \in \Omega$ and every $u \in \Omega$,

$$
\gamma D(x) - F(x, u) \geq (\gamma - 1)D(x) > 0
$$

we obtain from (4.4) that

$$
\begin{aligned}
\nu f(x, u)u &\leq -(\gamma D(x) - F(x, u)) + (\gamma D(x))^\nu ((\gamma - 1)D(x))^{1-\nu} \\
&\leq F(x, u) + \gamma^\nu (\gamma - 1)^{1-\nu} D(x).
\end{aligned}
$$

The lemma is proved. $\square$

Fix a C^∞-function $\overline{\vartheta} : \mathbb{R} \to [0, 1]$ with $\overline{\vartheta}(s) = 0$ for $s \in \,]-\infty, 1]$ and $\overline{\vartheta}(s) = 1$ for $s \in [2, \infty[$. Let

$$
\vartheta := \overline{\vartheta}^2.
$$

For $k \in \mathbb{N}$ let the functions $\overline{\vartheta}_k : \mathbb{R}^N \to \mathbb{R}$ and $\vartheta_k : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$
\overline{\vartheta}_k(x) = \overline{\vartheta}(|x|^2 / k^2) \text{ and } \vartheta_k(x) = \vartheta(|x|^2 / k^2), \quad x \in \mathbb{R}^N.
$$

Theorem 4.4. *Assume Hypothesis 4.2. Choose δ and $\nu \in \,]0, \infty[$ with*

$$
(4.5) \qquad \nu \leq \min(1, \overline{\mu}/2), \ \lambda_1 - \delta\alpha_1 > 0 \ \text{and} \ \alpha_0 - 2\delta\varepsilon \geq 0.
$$

Under these hypotheses, there is a constant $c' \in [0, \infty[$ and for every $R \in [0, \infty[$ there are constants $M' = M'(R)$, $c_k = c_k(R) \in [0, \infty[$, $k \in \mathbb{N}$ with $c_k \to 0$ for $k \to \infty$ and such that for every $\tau_0 \in [0, \infty[$ and every solution $z(\cdot)$ of π_f on $I = [0, \tau_0]$ with $|z(0)|_Z \leq R$

$$
(4.6)
$$
$$
\int_\Omega \big((\varepsilon/2)|z_2(t)(x)|^2 + (A(x)\nabla z_1(t)(x)) \cdot \nabla z_1(t)(x) + (\beta(x) - \delta\alpha(x))|z_1(t)(x)|^2 \big)\,\mathrm{d}x
$$
$$
\leq c' + M' e^{-2\delta\nu t}, \quad t \in I.
$$

If $|z(t)|_Z \leq R$ for $t \in I$, then

(4.7)
$$\int_\Omega \vartheta_k(x)\big((\varepsilon/2)|z_2(t)(x)|^2 + (A(x)\nabla z_1(t)(x)) \cdot \nabla z_1(t)(x)$$
$$+ (\beta(x) - \delta\alpha(x))|z_1(t)(x)|^2\big)\,\mathrm{d}x$$
$$\leq c_k + M'e^{-2\delta\nu t}, \quad k \in \mathbb{N},\ t \in I.$$

Lemma 4.5. *Assume the hypotheses of Theorem 4.4. Let $\overline{\gamma}$, γ, $V = V_\gamma$ and $V^* = V_\gamma^*$ be as in Proposition 4.1. For all $z \in Z$ and $x \in \Omega$ define $s(z)(x) = s_{\overline{\gamma}}(z)(x)$ by*

$$s(z)(x) = -2\overline{\gamma}(x)z_1(x)(A(x)\nabla\overline{\gamma}(x)) \cdot \nabla z_1(x) - |z_1(x)|^2(A(x)\nabla\overline{\gamma}(x)) \cdot \overline{\gamma}(x).$$

Given $\tau_0 \in [0, \infty[$ and a solution $z(\cdot)$ of π_f on $I = [0, \tau_0]$, define

$$\eta(t) = \eta_\gamma(t) = V_\gamma(z(t)) - V_\gamma^*(z(t)),\ t \in I.$$

Then

(4.8)
$$\eta'(t) + 2\delta\nu\eta(t) \leq 2\delta(\overline{\mu} - \nu)\int_\Omega \gamma(x)c(x)\,\mathrm{d}x$$
$$- \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma) \cdot \nabla z_1\,\mathrm{d}x - \delta(1 - \nu)\int_\Omega s_{\overline{\gamma}}(z(t))(x)\,\mathrm{d}x,\ t \in I.$$

Proof. It is clear that, for all $z \in Z$ and $x \in \Omega$

$$|\overline{\gamma}(x)|^2(A(x)\nabla z_1(x)) \cdot \nabla z_1(x) = (A(x)\nabla(\overline{\gamma}z_1)(x)) \cdot \nabla(\overline{\gamma}z_1)(x) + s(z)(x).$$

Thus, by the definition of V,

$$2V(z) \geq \int_\Omega \gamma(x)\big((A(x)\nabla z_1(x)) \cdot \nabla z_1(x) + (\beta(x) - \delta\alpha(x))|z_1(x)|^2\big)\,\mathrm{d}x$$
$$= \int_\Omega \big((A(x)\nabla(\overline{\gamma}z_1)(x)) \cdot \nabla(\overline{\gamma}z_1)(x) + (\beta(x) - \delta\alpha(x))|\overline{\gamma}(x)z_1(x)|^2\big)\,\mathrm{d}x$$
$$+ \int_\Omega s_{\overline{\gamma}}(z)(x)\,\mathrm{d}x \geq (\lambda_1 - \delta\alpha_1)|\overline{\gamma}z_1|_{L^2}^2 + \int_\Omega s_{\overline{\gamma}}(z)(x)\,\mathrm{d}x \geq \int_\Omega s_{\overline{\gamma}}(z)(x)\,\mathrm{d}x.$$

Hence

$$(V \circ z)'(t) + 2\delta\nu(V \circ z)(t) = (V \circ z)'(t) + 2\delta(V \circ z)(t) - \delta(1 - \nu)2V(z(t))$$
$$\leq (V \circ z)'(t) + 2\delta(V \circ z)(t) - \delta(1 - \nu)\int_\Omega s_{\overline{\gamma}}(z(t))(x)\,\mathrm{d}x.$$

It follows that

$$(V \circ z)'(t) + 2\delta\nu(V \circ z)(t) + \delta(1 - \nu) \int_\Omega s_{\overline{\gamma}}(z(t))(x) \, \mathrm{d}x$$

$$\leq (V \circ z)'(t) + 2\delta(V \circ z)(t)$$

$$\leq (2\delta\varepsilon - \alpha_0) \int_\Omega \gamma(x)(\delta z_1 + z_2)^2 \, \mathrm{d}x + \int_\Omega \gamma(x)(\delta z_1 + z_2) f(x, z_1(t)(x)) \, \mathrm{d}x$$

$$- \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma) \cdot \nabla z_1 \, \mathrm{d}x \leq \delta \int_\Omega \gamma(x) z_1 f(x, z_1(t)(x)) \, \mathrm{d}x$$

$$+ \int_\Omega \gamma(x) z_2 f(x, z_1(t)(x)) \, \mathrm{d}x - \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma) \cdot \nabla z_1 \, \mathrm{d}x$$

$$\leq \delta\overline{\mu} \int_\Omega \gamma(x)(F(x, z_1(t)(x)) + c(x)) \, \mathrm{d}x - 2\delta\nu(V^* \circ z)(t)$$

$$+ 2\delta\nu(V^* \circ z)(t) + (V^* \circ z)'(t) - \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma) \cdot \nabla z_1 \, \mathrm{d}x =: S^*$$

Now

$$S^* = \delta(\overline{\mu} - 2\nu) \int_\Omega \gamma(x) F(x, z_1(t)(x)) \, \mathrm{d}x + \delta\overline{\mu} \int_\Omega \gamma(x) c(x) \, \mathrm{d}x$$

$$+ 2\delta\nu(V^* \circ z)(t) + (V^* \circ z)'(t) - \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma) \cdot \nabla z_1 \, \mathrm{d}x$$

$$\leq \delta(2\overline{\mu} - 2\nu) \int_\Omega \gamma(x) c(x) \, \mathrm{d}x$$

$$+ 2\delta\nu(V^* \circ z)(t) + (V^* \circ z)'(t) - \int_\Omega (\delta z_1 + z_2)(A\nabla\gamma) \cdot \nabla z_1 \, \mathrm{d}x.$$

This immediately implies (4.8) and proves the lemma. □

Proof of Theorem 4.4. Let $\tau_0 \in [0, \infty[$ be arbitrary and $z(\cdot)$ of π_f be an arbitrary solution of π_f on $I = [0, \tau_0]$ with $|z(0)|_Z \leq R$. Set $\overline{\gamma} = \gamma \equiv 1$. Then $s_{\overline{\gamma}}(z(t)) \equiv 0$. Thus Lemma 4.5 implies that

$$(4.9) \qquad\qquad\qquad \eta'_\gamma + 2\delta\nu\eta_\gamma \leq \overline{c}.$$

where $\overline{c} = 2\delta(\overline{\mu} - \nu) \int_\Omega \gamma(x) c(x) \, \mathrm{d}x$. Differentiating the function $t \mapsto \eta_\gamma(t) e^{2\delta\nu t}$ and using (4.9) we obtain

$$(4.10) \qquad\qquad \eta_\gamma(t) \leq (1/(2\delta\nu))\overline{c}[1 - e^{-2\delta\nu t}] + \eta_\gamma(0) e^{-2\delta\nu t}, \quad t \in I.$$

Our assumptions imply that there is a continuous imbedding $H_0^1(\Omega) \to L^{\overline{\rho}+2}(\Omega)$ with an imbedding constant C_2. Let L_β, resp. L_a be bounds on the operators from $H_0^1(\Omega)$ to $L^2(\Omega)$ given by the assignments $u \mapsto |\beta|^{1/2} u$, resp. $u \mapsto |a|^{1/2} u$. Now a simple calculation using Proposition 3.11 shows that

$$(4.11) \qquad \begin{aligned} |\eta_\gamma(0)| &\leq (1/2)(2\delta^2\varepsilon R^2 + 2\varepsilon R^2 + a_1 R^2 + (L_\beta^2 + \delta^2\varepsilon)R^2) \\ &\quad + \overline{C}(L_a^2 R^2/2 + (C_2)^{\overline{\rho}+2} R^{\overline{\rho}+2}/(\overline{\rho} + 2)) + R\overline{\tau} =: \overline{M}. \end{aligned}$$

The definitions of V_γ and V_γ^* and our assumption on F now imply that, for $t \in I$,

$$
\begin{aligned}
(4.12) \quad (1/2) \int_\Omega & \big(\varepsilon |\delta z_1(t)(x) + z_2(t)(x)|^2 + (A(x)\nabla z_1(t)(x)) \cdot \nabla z_1(t)(x) \\
& + (\beta(x) - \delta\alpha(x) + \delta^2\varepsilon)|z_1(t)(x)|^2\big)\, \mathrm{d}x \\
& \leq (1/(2\delta\nu))\overline{c}[1 - e^{-2\delta\nu t}] + \overline{M}e^{-2\delta\nu t} + V_\gamma^*(z(t)) \\
& \leq (1/(2\delta\nu))\overline{c}[1 - e^{-2\delta\nu t}] + \overline{M}e^{-2\delta\nu t} + \int_\Omega c(x)\, \mathrm{d}x.
\end{aligned}
$$

Now, for a_1, $a_2 \in \mathbb{R}$ we have

$$
|a_1|^2 = |(a_1 + a_2) + (-a_2)|^2 \leq 2(|a_1 + a_2|^2 + |a_2|^2)
$$

so

$$
|a_1 + a_2|^2 \geq (1/2)|a_1|^2 - |a_2|^2
$$

and thus setting $a_1 = z_2(t)(x)$ and $a_2 = \delta z_1(t)(x)$ in (4.12) we obtain

$$
\begin{aligned}
(4.13) \quad (1/2) \int_\Omega & \big((\varepsilon/2)|z_2(t)(x)|^2 + (A(x)\nabla z_1(t)(x)) \cdot \nabla z_1(t)(x) \\
& + (\beta(x) - \delta\alpha(x))|z_1(t)(x)|^2\big)\, \mathrm{d}x \\
& \leq (1/(2\delta\nu))\overline{c}[1 - e^{-2\delta\nu t}] + \overline{M}e^{-2\delta\nu t} + \int_\Omega c(x)\, \mathrm{d}x.
\end{aligned}
$$

Setting $c' = 2((1/(2\delta\nu))\overline{c} + \int_\Omega c(x)\, \mathrm{d}x)$ and $M' = 2\overline{M}$ we obtain (4.6).

Assume now that $|z(t)|_Z \leq R$ for all $t \in I$. Let $k \in \mathbb{N}$ be arbitrary and set $V_k = V_{\gamma_k}$, $V_k^* = V_{\gamma_k}^*$, $s_k(z)(x) = s_{\overline{\gamma}_k}(z)(x)$ and $\eta_k(t) = \eta_{\gamma_k}$, where $\overline{\gamma}_k = \overline{\vartheta}_k$ and $\gamma_k = \vartheta_k$. Since $\nabla\vartheta_k(x) = (2/k^2)\vartheta'(|x|^2/k^2)x$ and $\nabla\overline{\vartheta}_k(x) = (2/k^2)\overline{\vartheta}'(|x|^2/k^2)x$ we have

$$
(4.14) \qquad \sup_{x \in \Omega} |\nabla\vartheta_k(x)| \leq C_\vartheta/k \quad \text{and} \quad \sup_{x \in \Omega} |\nabla\overline{\vartheta}_k(x)| \leq C_{\overline{\vartheta}}/k
$$

where $C_\vartheta = 2\sqrt{2}\sup_{y \in \mathbb{R}} |\vartheta'(y)|$ and $C_{\overline{\vartheta}} = 2\sqrt{2}\sup_{y \in \mathbb{R}} |\overline{\vartheta}'(y)|$.

We thus obtain

$$
(4.15) \qquad -\int_\Omega (\delta z_1 + z_2)(A\nabla\vartheta_k) \cdot \nabla z_1\, \mathrm{d}x \leq a_1(C_\vartheta/k)(\delta R + R)R
$$

and

$$
(4.16) \qquad -\delta(1 - \nu)\int_\Omega s_k(z(t))(x)\, \mathrm{d}x \leq a_1\delta(1 - \nu)(2C_{\overline{\vartheta}}/k + C_{\overline{\vartheta}}^2/k^2)R^2.
$$

Set

$$\text{(4.17)} \quad \begin{aligned} \xi_k &= 2\delta(\overline{\mu} - \nu) \int_{\{x \in \Omega \,|\, |x| \geq k\}} |c(x)| \, \mathrm{d}x \\ &+ a_1 (C_\vartheta/k)(\delta R + R)R + a_1 \delta(1 - \nu)(2C_{\overline{\vartheta}}/k + C_{\overline{\vartheta}}^2/k^2)R^2. \end{aligned}$$

Using Lemma 4.5 we thus have that

$$\text{(4.18)} \qquad \eta_k' + 2\delta\nu\eta_k \leq \xi_k, \quad k \in \mathbb{N}.$$

Differentiating the function $t \mapsto \eta_k(t)e^{2\delta\nu t}$ and using (4.18) we obtain

$$\text{(4.19)} \qquad \eta_k(t) \leq (1/(2\delta\nu))\xi_k[1 - e^{-2\delta\nu t}] + \eta_k(0)e^{-2\delta\nu t}, \quad t \in I.$$

We have

$$\text{(4.20)} \qquad |\eta_k(0)| \leq \overline{M}$$

where $\overline{M}$ is as in (4.11). Using our assumptions on ϑ we obtain

$$\text{(4.21)} \qquad V^*(z(t)) \leq \int_\Omega \vartheta_k(x)c(x)\,\mathrm{d}x \leq \int_{\{x \in \Omega \,|\, |x| \geq k\}} c(x)\,\mathrm{d}x =: \zeta_k, \quad t \in I.$$

It follows that, for $t \in I$,

$$\text{(4.22)} \quad \begin{aligned} (1/2)\int_\Omega \vartheta_k(x)\big(\varepsilon|\delta z_1(t)(x) &+ z_2(t)(x)|^2 + (A(x)\nabla z_1(t)(x)) \cdot \nabla z_1(t)(x) \\ &+ (\beta(x) - \delta\alpha(x) + \delta^2\varepsilon)|z_1(t)(x)|^2\big)\,\mathrm{d}x \leq (1/(2\delta\nu))\xi_k + \overline{M}e^{-2\delta\nu t} + \zeta_k. \end{aligned}$$

As before, this implies that

$$\text{(4.23)} \quad \begin{aligned} (1/2)\int_\Omega \vartheta_k(x)\big((\varepsilon/2)|z_2(t)(x)|^2 &+ (A(x)\nabla z_1(t)(x)) \cdot \nabla z_1(t)(x) \\ &+ (\beta(x) - \delta\alpha(x))|z_1(t)(x)|^2\big)\,\mathrm{d}x \\ &\leq (1/(2\delta\nu))\xi_k + \overline{M}e^{-2\delta\nu t} + \zeta_k. \end{aligned}$$

Setting $M' = 2\overline{M}$ and $c_k = 2((1/(2\delta\nu))\xi_k + \zeta_k)$, $k \in \mathbb{N}$ we obtain (4.7). The theorem is proved. $\square$

Theorem 4.6. *Assume Hypothesis* 4.2. *Then π_f is a global semiflow. Moreover, there is a constant $C_{\pi_f} \in [0, \infty[$ with the property that for every z_0 there is a $t_{z_0} \in [0, \infty[$ such that $|z_0\pi_f t|_Z \leq C_{\pi_f}$ for all $t \in [t_{z_0}, \infty[$. Furthermore, every bounded subset of Z is ultimately bounded (rel. to π_f).*

Proof. Using the first part of Theorem 4.4 together with Lemma 3.4 (with $\kappa = \delta\alpha_1$) we conclude that for every $z_0 \in Z$ there is a constant $C_{z_0} \in [0, \infty[$ such that $|z_0\pi_f t|_Z \leq C_{z_0}$ for $t \in [0, \omega_{z_0}[$. Since π_f does not explode in bounded subsets of Z, this implies that $\omega_{z_0} = \infty$, so π_f is a global semiflow. Similar arguments prove the other assertions of the theorem. $\square$

Now consider the following alternative hypotheses:

Hypothesis 4.7. $\overline{\rho}$ is subcritical and $\tilde{a} \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for some $r \in \mathbb{R}$ with $r > \max(N, 2)$.

Hypothesis 4.8. $\overline{\rho}$ is critical, $a \in L^r(\Omega) + L^\infty(\Omega)$ for some $r \in [N, \infty[$ and (3.16) is satisfied.

Lemma 4.9. Let $\tilde{N}$ be an arbitrary ultimately bounded set in $Z = H^1_0(\Omega) \times L^2(\Omega)$ (relative to π_f), $(z_n)_n$ be an arbitrary sequence in $\tilde{N}$ and $(t_n)_n$ be a sequence in $[0, \infty[$ with $t_n \to \infty$.

    (1) if Hypothesis 4.7 holds, then the sequence $(z_n \pi_f t_n)_n$ has a subsequence which converges in Z.

    (2) if Hypothesis 4.8 holds, then $(z_n \pi_f t_n)_n$ has a subsequence which converges in $Y = L^2(\Omega) \times H^{-1}(\Omega)$.

Proof. There is a $t_{\tilde{N}}$ and an $R \in [0, \infty[$ such that $|z\pi_f t|_Z \le R$ for all $z \in \tilde{N}$ and all $t \in [t_{\tilde{N}}, \infty[$. We may assume that $t_n \ge t_{\tilde{N}}$ and therefore, replacing z_n by $z_n \pi_f t_{\tilde{N}}$ and t_n by $t_n - t_{\tilde{N}}$ we may assume that $|z_n \pi_f t|_Z \le R$ for all $n \in \mathbb{N}$ and $t \in [0, t_n]$. For $n \in \mathbb{N}$ and $t \in [0, t_n]$ let $u_n(t)$ be the first component of $z_n \pi_f t$. Let $\tau_0 \in]0, \infty[$ be arbitrary to be determined later. Then there an $n_0(\tau_0) \in \mathbb{N}$ such that $t_n \ge 2\tau_0$ for all $n \in \mathbb{N}$ with $n \ge n_0(\tau_0)$. For such n we have

$$z_n \pi_f t_n = T(\tau_0) z_n \pi_f (t_n - \tau_0)$$
$$+ \int_0^{\tau_0} T(\tau_0 - s)(0, (1/\varepsilon)(\hat{f}(u_n(t_n - \tau_0 + s)) - \hat{f}((1 - \overline{\vartheta}_k) u_n(t_n - \tau_0 + s)))) \, \mathrm{d}s$$
$$+ \int_0^{\tau_0} T(\tau_0 - s)(0, (1/\varepsilon)\hat{f}((1 - \overline{\vartheta}_k) u_n(t_n - \tau_0 + s)) \, \mathrm{d}s$$

We have

(4.24) $$|T(\tau_0) z_n \pi_f (t_n - \tau_0)|_Z \le M e^{-\mu \tau_0} R, \quad n \ge n_0(\tau_0).$$

Since $\sup_{k \in \mathbb{N}} |\overline{\vartheta}_k|_{W^{1,\infty}(\mathbb{R}^N)} < \infty$ it follows from Lemma 3.9 that

$$\sup_{k, n \in \mathbb{N}} \sup_{t \in [0, t_N]} (|u_n(t)|_{H^1_0} + |(1 - \overline{\vartheta}_k) u_n(t)|_{H^1_0}) < \infty.$$

It follows from our hypotheses and from Proposition 3.12 that there is an $L \in]0, \infty[$ such that for all $k \in \mathbb{N}$, $n \in \mathbb{N}$ and $t \in [0, t_n]$

$$|\hat{f}(u_n(t)) - \hat{f}((1 - \overline{\vartheta}_k) u_n(t))|_{L^2} \le L|\overline{\vartheta}_k u_n(t)|_{H^1_0}.$$

This implies that

(4.25)
$$|\int_0^{\tau_0} T(\tau_0 - s)(0, (1/\varepsilon)(\hat{f}(u_n(t_n - \tau_0 + s)) - \hat{f}((1 - \overline{\vartheta}_k) u_n(t_n - \tau_0 + s)))) \, \mathrm{d}s|_Z$$

$$\le \sup_{s \in [0, \tau_0]} |\overline{\vartheta}_k u_n(t_n - \tau_0 + s)|_{H^1_0} (1/\varepsilon) LM \int_0^{\tau_0} e^{-\mu(\tau_0 - s)} \, \mathrm{d}s$$

$$\le (LM/(\mu\varepsilon)) \sup_{s \in [0, \tau_0]} |\overline{\vartheta}_k u_n(t_n - \tau_0 + s)|_{H^1_0}, \quad n \ge n_0(\tau_0).$$

Now use Lemma 3.4 with $\kappa = \delta\alpha_1$. Let $c > 0$ be as in that Lemma. It follows that, for k, $n \in \mathbb{N}$ and $t \in [0, t_n]$

$$(4.26) \quad \begin{aligned} c|\overline{\vartheta}_k u_n(t)|^2_{H^1_0} &\leq \langle A\nabla(\overline{\vartheta}_k u_n(t)), \nabla(\overline{\vartheta}_k u_n(t))\rangle + \langle \beta\overline{\vartheta}_k u_n(t), \overline{\vartheta}_k u_n(t)\rangle \\ &\quad - \delta\alpha_1\langle \overline{\vartheta}_k u_n(t), \overline{\vartheta}_k u_n(t)\rangle \\ &\leq \langle A\nabla(\overline{\vartheta}_k u_n(t)), \nabla(\overline{\vartheta}_k u_n(t))\rangle + \langle \beta\overline{\vartheta}_k u_n(t), \overline{\vartheta}_k u_n(t)\rangle \\ &\quad - \delta\langle \alpha\overline{\vartheta}_k u_n(t), \overline{\vartheta}_k u_n(t)\rangle \\ &= \int_\Omega \vartheta_k(x)\big(\langle A\nabla u_n(t), \nabla u_n(t)\rangle + (\beta(x) - \delta\alpha(x))|u_n(t)(x)|^2\big)\,\mathrm{d}x \\ &\quad + 2\langle \overline{\vartheta}_k A\nabla u_n(t), u_n(t)\nabla\overline{\vartheta}_k\rangle + \langle u_n(t)A\nabla\overline{\vartheta}_k, u_n(t)\nabla\overline{\vartheta}_k\rangle \\ &\leq c_k + M'e^{-2\delta\nu t} + a_1(2C_{\overline{\vartheta}}/k + C^2_{\overline{\vartheta}}/k^2)R^2 \end{aligned}$$

Now, if $n \geq n_0(\tau_0)$ and $s \in [0, \tau_0]$ then $t = t_n - \tau_0 + s \geq \tau_0$ so (4.26) implies that

$$\sup_{n \geq n_0(\tau_0)} \sup_{s \in [0,\tau_0]} |\overline{\vartheta}_k u_n(t_n - \tau_0 + s)|_{H^1_0} \to 0$$

for $k \to \infty$ and $\tau_0 \to \infty$. It follows that the right hand sides of (4.24) and (4.25) can be made as small as we wish, by taking $k \in \mathbb{N}$ and $\tau_0 > 0$ sufficiently large. Therefore, a standard argument using Kuratowski measure of noncompactness implies that the sequence $(z_n\pi_f t_n)_n$ has a subsequence which converges in Z (resp. in Y) provided we can prove that, for every $k \in \mathbb{N}$ and $\tau_0 \in {]0, \infty[}$ the set

$$K_0 := \{\, T(\tau_0 - s)(0, (1/\varepsilon)\hat{f}((1 - \overline{\vartheta}_k)u_n(t_n - \tau_0 + s))) \mid n \geq n_0(\tau_0),\ s \in [0, \tau_0] \,\}$$

is relatively compact in Z (resp. in Y).

Let $(z_l)_l$ be a sequence in K_0. It follows that for every $l \in \mathbb{N}$ there are $n_l \in \mathbb{N}$ $s_l \in [0, \tau_0]$ with $z_l = T(\tau_0 - s_l)(0, (1/\varepsilon)\hat{f}(v_l))$ where $v_l = (1 - \overline{\vartheta}_k)u_{n_l}(t_{n_l} - \tau_0 + s_l)$. By choosing subsequences if necessary we may assume that $s_l \to s_\infty$ for some $s_\infty \in [0, \tau_0]$. By Proposition 3.10 $(v_l)_l$ is compact in $L^s(\Omega)$ for each $s \in [2, \infty[$ such that $s \in [2, 2^*[$ if $N \geq 3$.

First suppose that Hypothesis 4.7 holds. Then $s \in [2, 2^*[$ for $s \in \{2r/(r - 2), 2(\overline{\rho} + 1)\}$. Taking subsequences if necessary, we may thus assume that there is a $v \in H^1_0(\Omega)$ such that $(v_l)_l$ converges to v weakly in $H^1_0(\Omega)$ and strongly in $L^s(\Omega)$ for $s \in \{2r/(r - 2), 2(\overline{\rho} + 1)\}$. Moreover, whenever $x \in \Omega$ and $|x| \geq \sqrt{2}k$ then $v_l(x) = 0$ for all $l \in \mathbb{N}$, and so we may assume that $v(x) = 0$. Thus

$$(4.27) \qquad a(x)(v_l(x) - v(x)) = a_1(x)(v_l(x) - v(x)), \quad l \in \mathbb{N},\ x \in \Omega$$

where $a_1 \colon \mathbb{R}^N \to \mathbb{R}$ is defined by $a_1(x) = \tilde{a}(x)$ if $x \in \overline{\Omega}$ and $|x| \leq \sqrt{2}k$ and $a_1(x) = 0$ otherwise. Note that $a_1 \in L^r(\mathbb{R}^N)$ so the map $L^{2r/(r-2)}(\Omega) \to L^2(\Omega)$, $h \mapsto a_1 h$ is defined, linear and bounded. Now (3.24) and (4.27) imply that $|\hat{f}(v_l) - \hat{f}(v)|_{L^2} \to 0$ as $l \to \infty$. This clearly implies that $|z_l - T(\tau_0 - s_\infty)(0, (1/\varepsilon)\hat{f}(v))|_Z \to 0$ as $l \to \infty$.

Now suppose Hypothesis 4.8. Then $2 \in [2, 2^*[$ for $N \geq 3$. Taking subsequences if necessary, we may thus assume that there is a $v \in H_0^1(\Omega)$ such that $(v_l)_l$ converges to v weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Using (3.28) we obtain that $|\hat{f}(v_l) - \hat{f}(v)|_{H^{-1}} \to 0$ as $l \to \infty$. Proposition 3.8 now implies that $|z_l - T(\tau_0 - s_\infty)(0, (1/\varepsilon)\hat{f}(v))|_Y \to 0$ as $l \to \infty$. The lemma is proved. $\square$

We can now prove the first main result of this paper.

Theorem 4.10. *Assume Hypotheses 4.2 and 4.7. Then π_f is a global semiflow and it has a global attractor.*

Proof. This is an immediate consequence of Theorem 4.6, Lemma 4.9 and Proposition 2.1. $\square$

We will now treat the critical case.

Proposition 4.11. *Assume Hypotheses 4.2 and 4.8. Let $C_6 \in [0, \infty[$ be arbitrary. Then there is a constant $C_7 \in [0, \infty[$ such that whenever $t \in [0, \infty[$ and z_1 and $z_2 \in Z$ are such that $|z_1|_Z \leq C_6$ and $|z_2|_Z \leq C_6$ then*

$$|z_1\pi_f t - z_2\pi_f t|_Y \leq C_7 e^{C_7 t}|z_1 - z_2|_Y.$$

Proof. By Theorem 4.4 and Lemma 3.4 there is a constant $C_8 \in [0, \infty[$ such that whenever $z \in Z$ and $|z|_Z \leq C_6$ then $|z\pi_f t|_Z \leq C_8$ for all $t \in [0, \infty[$. By (3.28) we now obtain a constant $C_9 \in [0, \infty[$ such that $|\hat{f}(u_1) - \hat{f}(u_2)|_{H^{-1}} \leq C_9|u_1 - u_2|_{L^2}$ for all $z_1 = (u_1, v_1)$, $z_2 = (u_2, v_2) \in Z$ with $|z_1|_Z \leq C_8$ and $|z_2|_Z \leq C_8$. Now Proposition 3.8, the variation-of-constants formula and Gronwall's lemma complete the proof. $\square$

Theorem 4.12. *Assume Hypotheses 4.2 and 4.8. Then π_f is asymptotically compact.*

Proof. We use an ingenious method due to J. Ball, cf. [5, 18, 20].

Let $\tilde{N}$ be a π_f-ultimately bounded subset of Z. Then there is a $t_{\tilde{N}} \in [0, \infty[$ and a $C_{10} \in [0, \infty[$ such that $|z\pi_f t| \leq C_{10}$ whenever $z \in \tilde{N}$ and $t \geq t_{\tilde{N}}$. Let $(z_n)_n$ be an arbitrary sequence in $\tilde{N}$ and $(t_n)_n$ be an arbitrary sequence in $[0, \infty[$ with $t_n \to \infty$ as $n \to \infty$. We must prove that a subsequence of $(z_n\pi_f t_n)_n$ converges strongly in Z. Now using Lemma 4.9 and Cantor's diagonal procedure we see that there is a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and for every $l \in \mathbb{Z}$ with $l \geq 0$ there are a $k_0(l) \in \mathbb{N}$ and a $w_l \in Z$ with $|w_l| \leq C_{10}$ such that $t_{n_k} - l \geq t_{\tilde{N}}$ for $k \geq k_0(l)$ and the sequence $(z_{n_k}\pi_f(t_{n_k} - l))_{k \geq k_0(l)}$ converges to w_l weakly in Z and strongly in Y. By Proposition 4.11, for every $l \in \mathbb{N}$ and $t \in [0, \infty[$,

$$(4.28) \qquad (z_{n_k}\pi_f(t_{n_k} - l))\pi_f t \to w_l\pi_f t, \text{ as } k \to \infty, \text{ strongly in } Y.$$

This shows that $w_l\pi_f l = w_0$ for all $l \in \mathbb{N}$. Now define the function $\mathcal{F}: Z \to \mathbb{R}$ by

$$\mathcal{F}(z) = V(z) - V^*(z), \quad z \in Z$$

where V and V^* are as in Proposition 4.1 with $\gamma \equiv 0$ and $\delta \in \,]0, \infty[$ such that $\lambda - \delta\alpha_1 > 0$ and $\alpha_0 - 2\delta\varepsilon \geq 0$. Using (3.25) we see that there is a constant $C_{11} \in [0, \infty[$ such that

$$\sup_{z \in Z, |z|_Z \leq C_{10}} |\mathcal{F}(z)| \leq C_{11}.$$

Note that $\Psi \colon Z \to Z$, $(u, v) \mapsto (u, \delta u + v)$, is an isomorphism of normed spaces. Thus

$$[(u_1, v_1), (u_2, v_2)] := \varepsilon\langle \delta u_1 + v_1, \delta u_2 + v_2\rangle + \langle A\nabla u_1, u_2\rangle + \langle(\beta - \delta\alpha + \delta^2\varepsilon)u_1, u_2\rangle$$

defines a scalar product on Z whose norm $z \mapsto \|z\| := \sqrt{[z, z]}$ is equivalent to the usual norm on Z. Note that $\mathcal{F}(z) = \|z\|^2 - V^*(z)$ for $z \in Z$.

Let $\zeta = (\zeta_1, \zeta_2) \colon [0, \infty[\,\to Z$ be an arbitrary solution of π_f. Proposition 4.1 implies that the function $\mathcal{F} \circ \zeta$ is continuously differentiable and for every $t \in [0, \infty[$

$$(\mathcal{F} \circ \zeta)'(t) + 2\delta\mathcal{F}(\zeta(t)) = \int_\Omega (2\delta\varepsilon - \alpha(x))(\delta\zeta_1(t)(x) + \zeta_2(t)(x))^2 \,\mathrm{d}x$$

$$+ \int_\Omega \delta\zeta_1(t)(x)f(x, \zeta_1(t)(x)) \,\mathrm{d}x - 2\delta \int_\Omega F(x, \zeta_1(t)(x)) \,\mathrm{d}x.$$

It follows that for every $t \in [0, \infty[$
(4.29)
$$\mathcal{F}(\zeta(t)) = e^{-2\delta t}\mathcal{F}(\zeta(0))$$

$$+ \int_0^t e^{-2\delta(t-s)} \left(\int_\Omega (2\delta\varepsilon - \alpha(x))(\delta\zeta_1(s)(x) + \zeta_2(s)(x))^2 \,\mathrm{d}x \right) \,\mathrm{d}s$$

$$+ \int_0^t e^{-2\delta(t-s)} \left(\int_\Omega \delta\zeta_1(s)(x)f(x, \zeta_1(s)(x)) \,\mathrm{d}x - 2\delta \int_\Omega F(x, \zeta_1(s)(x)) \,\mathrm{d}x \right) \,\mathrm{d}s.$$

Fix $l \in \mathbb{N}$ and, for $k \geq k_0(l)$, let $\zeta_k(t) = (z_{n_k}\pi_f(t_{n_k} - l))\pi_f t$ and $\zeta(t) = w_l\pi_f t$ for $t \in [0, \infty[$. Then (4.29) with $t = l$ implies that
(4.30)
$$\|z_{n_k}\pi_f(t_{n_k})\|^2 - V^*(z_{n_k}\pi_f(t_{n_k})) = e^{-2\delta l}\mathcal{F}(z_{n_k}\pi_f(t_{n_k} - l))$$

$$+ \int_0^l e^{-2\delta(l-s)} \left(\int_\Omega (2\delta\varepsilon - \alpha(x))(\delta\zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \,\mathrm{d}x \right) \,\mathrm{d}s$$

$$+ \int_0^l e^{-2\delta(l-s)} \left(\int_\Omega \delta\zeta_{k,1}(s)(x)f(x, \zeta_{k,1}(s)(x)) \,\mathrm{d}x - 2\delta \int_\Omega F(x, \zeta_{k,1}(s)(x)) \,\mathrm{d}x \right) \,\mathrm{d}s.$$

and
(4.31)
$$\|w_0\|^2 - V^*(w_0) = e^{-2\delta l}\mathcal{F}(w_l)$$

$$+ \int_0^l e^{-2\delta(l-s)} \left(\int_\Omega (2\delta\varepsilon - \alpha(x))(\delta\zeta_1(s)(x) + \zeta_2(s)(x))^2 \,\mathrm{d}x \right) \,\mathrm{d}s$$

$$+ \int_0^l e^{-2\delta(l-s)} \left(\int_\Omega \delta\zeta_1(s)(x)f(x, \zeta_1(s)(x)) \,\mathrm{d}x - 2\delta \int_\Omega F(x, \zeta_1(s)(x)) \,\mathrm{d}x \right) \,\mathrm{d}s.$$

Using (3.26) and (3.28) we see that

$$V^*(z_{n_k} \pi_f(t_{n_k})) \to V^*(w_0) \tag{4.32}$$

and
(4.33)
$$\int_0^l e^{-2\delta(l-s)} \left(\int_\Omega \delta\zeta_{k,1}(s)(x) f(x, \zeta_{k,1}(s)(x)) \, dx - 2\delta \int_\Omega F(x, \zeta_{k,1}(s)(x)) \, dx \right) ds$$
$$\to \int_0^l e^{-2\delta(l-s)} \left(\int_\Omega \delta\zeta_1(s)(x) f(x, \zeta_1(s)(x)) \, dx - 2\delta \int_\Omega F(x, \zeta_1(s)(x)) \, dx \right) ds$$

as $k \to \infty$. We claim that

(4.34)
$$\limsup_{k\to\infty} \int_0^l e^{-2\delta(l-s)} \left(\int_\Omega (2\delta\varepsilon - \alpha(x))(\delta\zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \, dx \right) ds$$
$$\le \int_0^l e^{-2\delta(l-s)} \left(\int_\Omega (2\delta\varepsilon - \alpha(x))(\delta\zeta_1(s)(x) + \zeta_2(s)(x))^2 \, dx \right) ds.$$

In fact, since $\alpha(x) - 2\delta\varepsilon \ge 0$ for all $x \in \Omega$ we have by Fatou's lemma
(4.35)
$$\limsup_{k\to\infty} \int_0^l e^{-2\delta(l-s)} \left(\int_\Omega (2\delta\varepsilon - \alpha(x))(\delta\zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \, dx \right) ds$$
$$= -\liminf_{k\to\infty} \int_0^l e^{-2\delta(l-s)} \left(\int_\Omega (\alpha(x) - 2\delta\varepsilon)(\delta\zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \, dx \right) ds$$
$$\le -\int_0^l e^{-2\delta(l-s)} \liminf_{k\to\infty} \left(\int_\Omega (\alpha(x) - 2\delta\varepsilon)(\delta\zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \, dx \right) ds$$

Let $s \in [0,l]$ be arbitrary. Since $((\zeta_{k,1}(s), \zeta_{k,2}(s)))_k$ converges to $(\zeta_1(s), \zeta_2(s))$ weakly in Z and Ψ is continuous, linear, hence weakly continuous, it follows that $((\zeta_{k,1}(s), \delta\zeta_{k,1}(s) + \zeta_{k,2}(s)))_k$ converges to $(\zeta_1(s), \delta\zeta_1(s) + \zeta_2(s))$ weakly in Z. It follows that for every $v \in L^2(\Omega)$

$$\langle v, \delta\zeta_{k,1}(s) + \zeta_{k,2}(s) \rangle \to \langle v, \delta\zeta_1(s) + \zeta_2(s) \rangle \text{ as } k \to \infty.$$

Taking $v = (\alpha - 2\delta\varepsilon)(\delta\zeta_1(s) + \delta\zeta_2(s))$ we thus obtain

$$|(\alpha - 2\delta\varepsilon)^{1/2}(\delta\zeta_1(s) + \delta\zeta_2(s))|_{L^2}^2$$
$$= \langle (\alpha - 2\delta\varepsilon)^{1/2}(\delta\zeta_1(s) + \delta\zeta_2(s)), (\alpha - 2\delta\varepsilon)^{1/2}(\delta\zeta_1(s) + \delta\zeta_2(s)) \rangle$$
$$= \lim_{k\to\infty} \langle (\alpha - 2\delta\varepsilon)^{1/2}(\delta\zeta_1(s) + \delta\zeta_2(s)), (\alpha - 2\delta\varepsilon)^{1/2}(\delta\zeta_{k,1}(s) + \delta\zeta_{k,2}(s)) \rangle$$
$$\le |(\alpha - 2\delta\varepsilon)^{1/2}(\delta\zeta_1(s) + \delta\zeta_2(s))|_{L^2} \liminf_{k\to\infty} |(\alpha - 2\delta\varepsilon)^{1/2}(\delta\zeta_{k,1}(s) + \delta\zeta_{k,2}(s))|_{L^2}$$

and so

$$(4.36) \quad \left(\int_\Omega (\alpha(x) - 2\delta\varepsilon)(\delta\zeta_1(s)(x) + \zeta_2(s)(x))^2 \, dx \right)$$
$$\leq \liminf_{k\to\infty} \left(\int_\Omega (\alpha(x) - 2\delta\varepsilon)(\delta\zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \, dx \right).$$

Inequalities (4.36) and (4.35) prove (4.34). Using (4.30), (4.31), (4.32), (4.33) and (4.34) we obtain

$$\limsup_{k\to\infty} \|z_{n_k}\pi_f(t_{n_k})\|^2 - V^*(w_l) \leq e^{-2\delta l} C_{11}$$
$$+ \int_0^l e^{-2\delta(l-s)} \left(\int_\Omega (2\delta\varepsilon - \alpha(x))(\delta\zeta_1(s)(x) + \zeta_2(s)(x))^2 \, dx \right) ds$$
$$+ \int_0^l e^{-2\delta(l-s)} \left(\int_\Omega \delta\zeta_1(s)(x) f(x, \zeta_1(s)(x)) \, dx - 2\delta \int_\Omega F(x, \zeta_1(s)(x)) \, dx \right) ds$$
$$= e^{-2\delta l} C_{11} + \|w_0\|^2 - V^*(w_0) - e^{-2\delta l}\mathcal{F}(w_l) \leq 2e^{-2\delta l} C_{11} + \|w_0\|^2 - V^*(w_0).$$

Thus for every $l \in \mathbb{N}$

$$\limsup_{k\to\infty} \|z_{n_k}\pi_f(t_{n_k})\|^2 \leq 2e^{-2\delta l} C_{11} + \|w_0\|^2$$

so

$$\limsup_{k\to\infty} \|z_{n_k}\pi_f(t_{n_k})\| \leq \|w_0\|.$$

Since $(z_{n_k}\pi_f(t_{n_k}))_k$ converges to w_0 weakly in $(Z, [\cdot, \cdot])$ we have

$$\liminf_{k\to\infty} \|z_{n_k}\pi_f(t_{n_k})\| \geq \|w_0\|.$$

Altogether we obtain

$$\lim_{k\to\infty} \|z_{n_k}\pi_f(t_{n_k})\| = \|w_0\|.$$

This implies that $(z_{n_k}\pi_f(t_{n_k}))_k$ converges to w_0 strongly in Z and completes the proof. $\square$

We can now prove the second main result of this paper.

Theorem 4.13. *Assume Hypotheses* 4.2 *and* 4.8. *Then* π_f *is a global semiflow and it has a global attractor.*

Proof. This is an immediate consequence of Theorem 4.6, Theorem 4.12 and Proposition 2.1. $\square$

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